*The chef refuses to work on an evening when there are **very few** customers and walks out. He will not return to work until an evening when there are **lots** of customers although he always comes-in on Friday because he gets paid then. We are interested in fraction of evenings that the chef is at the restaurant.*

Solution. In this example, the trial is whether the number of customers is *'lots'*, *'average'* or *'very few'* and it is repeated for each evening when the restaurant is open. Also, whether the chef is *'in'* or *'out'* depends not only on the outcome of this trial but also on his where-abouts the previous evening and whether it is a *Friday* or not. Here we wish to simulate the level of attendance at the restaurant on each evening. To determine whether he is *'in'* or *'out'* on that particular evening, we use the rules governing the chef's behaviour as stated in the question. Here the sequence of *'in's* and *out's* forms a simulated history of the system which helps us in determining the fraction of evenings for which the chef is *'in'*. Now we can proceed as follows :

Step 1. ***To simulate the number of customers each evening.***

Using the random digits and the given (prescribed) probability distribution, we simulate the number of customers each evening as given in the following Table 17.1.

Table 17.1.

Number of customers	Lots	Average	Very few
Probability	0.2	0.4	0.4
Digits	0, 1	2, 3, 4, 5	6, 7, 8, 9

Step 2. ***To generate attendance level.***

The attendance on a particular evening can be simulated by selecting a digit, from 0 to 9 inclusive, at random. Using the above table we can interpret the digit as state of the restaurant. On account of the number of digits assigned to each one, the outcomes in the simulation will have the same probability of occurrence as those in the real system.

Step 3. ***To determine whether the chef works or not.***

After generating an attendance level, we can determine whether the chef works or not by the rules provided in the question. The following table shows a specimen run of the simulation. (See Table 17.2).

The following specimen run was started with *'Monday'* and with the chef being *'In'*. Keeping the view in mind that we are trying to reproduce the long term behaviour of the restaurant, the initial conditions of the simulation will not affect the results we obtain.

Table 17.2. Specimen run

Day	Day of the week	Random digit	Number of customers	Chef'in' or 'out'	Number of days 'in' to date	Fraction of days'in' to date
1	Monday	3	Average	In	1	1.00
2	Tuesday	5	Average	In	2	1.00
3	Wednesday	1	Lots	In	3	1.00
4	Thursday	8	Very few	Out	3	0.75
5	Friday	6	Very few	In	4	0.80
6	Saturday	6	Very few	Out	4	0.67
7	Sunday	4	Average	Out	4	0.57
8	Monday	4	Average	Out	4	0.50
9	Tuesday	0	Lots	In	5	0.56
10	Wednesday					

Step 4. ***When to stop the simulation run.***

As far as determination of any average effects is concerned, the more time periods simulated the batter. However, in actual practice we require a criterion for deciding when the simulation should stop. In the specimen run we could continue until the fraction at the end of each line does not change very much. At this stage there is no point in going on further and we have the desired answer. In this example, it approaches to 0.49 and the ***chef is* 'in' *49% of all days***.

Example 6. *Mr. Sharma, the owner of small Grocery store wishes to evaluate his daily ordering policy for bread. His current rule is : order the amount demanded the previous day and he never runs out of stock. He* *purchases bread at the rate of Rs. 1.50 per bread and sells it for Rs. 1.75. The breads are ordered at the end of each day and are received the following morning.*

From past historical data the following distribution of demand for any day has been estimated.

Table 17.3.

Average Demand (D_n) (per day)	Probability
10	0.25
20	0.50
30	0.25

Mr. Sharma is considering the following two ordering policies :

Policy 1. *Order each day the amount of bread that was demanded pervious day, that is,* $O_n = D_{n-1}$

Policy 2. *Order each day the amount of bread that is equal to the average demand of previous two days.*

That is, $O_n = \frac{1}{2}(D_{n-1} + D_{n-2})$

Determine which policy will yield the highest profit ?

Solution. To compare the *Policy 1* with *Policy 2*, first we must define some rule of generating each day's demand. The rule could be stated in terms of total profits that may be stated as below :

Let S_n be the amount sold and P_n be the profit on a particular day (say, *n*th). Then,

$$S_n = D_n, \text{ if } D_n \le O_n \text{ ; } S_n = O_n, \text{ if } D_n > O_n \quad \text{and} \quad P_n = 0.25\, S_n$$

Now the general procedure to perform *Monte-Carlo simulation* involves the following steps :

Step 1. ***To determine a probability distribution for each random variable that requires analysis.***

First, to simulate *Policy* 1, we use the daily demand distribution to find a probability distribution as shown below :

Table 17.4.

Daily demand	Probability	Cumulative Probability	Random Numbers
10	0.25	0.25	00–24
20	0.50	0.75	25–74
30	0.25	1.00	75–99

Step 2. ***To determine cumulative probability distribution corresponding to each probability distribution.***

This is done in *column 3* of above table.

Step 3. ***To assign suitable set of random numbers for representing the value or the range of values of related variable(s).***

In order to generate the event in simulation in an unbiased manner, we assign random numbers to the events in the same proportion as their probability of occurrence.

Suppose we have 100 random numbers (0 to 99). Then, 25% of them (*i.e.*, 25 numbers) represent daily demand for 10 breads. So the numbers from 00 to 24 must be assigned to that variable. If we randomly select numbers large number of times in the range 0 to 99, then it will be expected that 25% of the times they will fall in the range 00 to 24. In the like manner, 50% of them must represent daily demand for 20 breads. So we must assign 50 random number, (25 to 74) to the second variable, and so on for the third variable.

Note. Here it should be noted that the assignment of random numbers follows the cumulative probability distribution of demand. It will be usually easier to assign the random numbers, if the cumulative probability distribution is found before making the assignments.

Step 4. ***To conduct the simulation experiment by random sampling for policy 1 and policy 2, respectively.***

If *Mr. Sharma* is interested in a ten days' simulation, then we take-up a sample of 10 random numbers from the table that will represent the sequence of samples. Here each random number represents the sample of demand.

Let D_0 = Expected demand at the beginning of simulation run (*i.e.*, on the 0th day)

$= 10 \times 0.25 + 20 \times 0.50 + 30 \times 0.25 = 20.$

The simulation computation for 10 day's period is given in the following table.

Table 17.5. Policy 1

Day (n)	Random Number	Demand (D_n)	Order Quantity ($O_n = D_{n-1}$)	Daily Profit (Rs.) ($P_n = 0.25\ S_n$)
0	—	20	—	—
1	40	20	20	5.00
2	19	10	20	2.50
3	87	30	10	2.50
4	83	30	30	7.50
5	73	20	30	5.00
6	84	30	20	5.00
7	29	20	30	5.00
8	09	10	20	2.50
9	02	10	10	2.50
10	20	10	10	2.50
	Total	190	200	Rs.40.00
	Daily Average	19	20	Rs. 4.00

For *Policy* 2, we assume that the expected demand for two days prior to the beginning of the simulation period will remain at the level of 20 units per day. Then, for *Policy* 2, the simulation computations are obtained as follows :

Table 17.6. Policy 2

Day (n)	Random Number	Demand (D_n)	Order Quantity [$O_n = ½(D_{n-1} + D_{n-1})$]	Daily Profit (Rs.) ($P_n = 0.25\ S_n$)
1	59	20	20	5.00
2	30	20	20	5.00
3	81	30	20	5.00
4	02	10	25	2.50
5	18	10	20	2.50
6	87	30	10	2.50
7	68	20	20	5.00
8	28	20	25	5.00
9	44	20	20	5.00
10	72	20	20	5.00
	Total	200	200	42.50
	Daily Average	20	20	4.25

Step 5. ***Repeat Step 4, until the desired number of simulation runs are generated.***

Since, *Mr. Sharma* is not interested in making more than one simulation run, so this step is not required in this example.

Step 6. ***Observation.***

From *Table* 17.5 and *Table* 17.6 it is observed that *policy* 2 contributes more profit in comparison to *policy* 1. But, due to small sample size of 10 days, *policy* 2 should not be adopted at once. It should be remembered that the larger is the sample size (number of runs), more accurate will be the decision derived by simulation technique.

Step 7. ***To design and implement a course of action and to maintain control.***

Lastly, observing the past behaviour of the system under study, an estimate of mean and standard deviation of the required outcome can be obtained. Then, these results will be more useful in designing, and implementing a course of action and proper control over the system's behaviour.

17.11. APPLICATIONS

Simulation has a large number of applications. For example, it can be used for learning about the operating characteristics of a new airplane by simulating flight conditions in a wind tunnel, on electronic or hydrolic analog models of production processes or economic systems, or on mathematical models of such real-life systems as inventory control, production scheduling, network analysis, and so on. It can also be used for planning military strategy, traffic control, management games and role playing, medical diagnosis, hospital emergency facilities, gambling and analysis, location analysis, *e.g.* determining optimal location for plants and warehouses, evaluation of industrial and commercial policies.

We now discuss a few applications in detail.

17.11-1. Applications to Inventory Control

For providing efficient service to the customers, it is necessary to choose to reorder point with proper consideration of demand during lead time. If the lead time and demand of inventory per unit time both are random variables, then the simulation technique can be applied to determine the effect of alternate inventory policies on a stochastic inventory system, *e.g.* different combinations of order quantity and reorder point. The basic approach would be to find the probability distribution of the input and output functions of the past data. Then, we run the inventory system artificially by generating the future observation on the assumptions of the same distributions. The method involves a good amount of computation. But, in simple problems it is possible to generate artificial samples for future with the help of random numbers and then the entire computations are done with the help of desk calculator. Of course, it becomes necessary to use electronic computers for solving more complex problems.

The technique is illustrated with the help of following simple inventory problems.

Example 7. *A book store wishes to carry 'Ramayana' in stock. Demand is probabilistic and replenishment of stock takes 2 days (i.e., if an order is placed on March 1, it will be delivered at the end of the day on March 3). The probabilities of demand are given below :*

Demand (daily) :	*0*	*1*	*2*	*3*	*4*
Probability :	*0.05*	*0.10*	*0.30*	*0.45*	*0.10*

Each time an order is placed, the store incurs an ordering cost of Rs. 10 per order. The store also incurs a carrying cost of Rs. 0.50 per book per day. The inventory carrying cost is calculated on the basis of stock at the time of each day.

The manager of the book store wishes to compare two options for his inventory decision.

A : Order 5 books when the inventory at the beginning of the day plus orders outstanding is less than 8 books.

B : Order 8 books when the inventory at the beginning of the day plus orders outstanding is less than 8.

Currently (beginning of the Ist day) the store has stock of 8 books plus 6 books ordered 2 days ago and expected to arrive next day.

Using Monte-Carlo simulation for 10 cycles, recommend which option the manager should choose.

The two digit random numbers are given below :

89, 34, 78, 63, 61, 81, 39, 16, 13, 73

Solution.

Demand	*Prob.*	*Cum. Prob.*	*Random Nos.*
0	0.05	0.05	00–04
1	0.10	0.15	05–14
2	0.30	0.45	15–44
3	0.45	0.90	45–89
4	0.10	1.00	90–99

Stock in hand = 8, and stock on order = 6 (expected next day).

Option A.

Random No.	Demand sales	Opt.Stock in hand	Receipt	Cl. stock in hand	Opt. stock on order	Order Qty.	Cl. stock on order
89	3	8	–	5	6	–	6
34	2	5	6	9	–	–	–
78	3	9	–	6	–	5	5
63	3	6	–	3	5	–	5
61	3	3	–	0	5	5	10
81	3	0	5	2	5	5	10
39	2	2	–	0	10	–	10
16	2	0	5	3	5	–	5
13	1	3	5	7	0	5	5
73	3	7	–	4	5	–	5

No. of orders = 4 (ordering cost) = 4 × 10 = Rs. 40.

Closing stock of 10 days = 39, carrying cost = 39 × 0.50 = 19.50

Cost for 10 days = 59.50.

Option B.

Sales	Opt. stock in hand	Receipt	Closing Stock on hand	Opt. Stock on order	Order Qty.	Closing Stock on order
3	8	–	5	6	–	6
2	5	6	9	–	–	–
3	9	–	6	–	8	8
3	6	–	3	8	–	8
3	3	–	0	8	–	8
3	3	8	5	–	8	8
2	5	–	3	8	–	8
2	3	–	1	8	–	8
1	1	8	8	–	–	–
3	8	–	5	–	8	8

No. of orders = 3, ordering cost = Rs. 30.

Closing stock of 10 days= 43, carrying cost = 45 × 0.50 = Rs. 22.50. Since option B has lower cost, manager should choose option B.

Example 8. *Consider an inventory situation in a manufacturing concern. If the number of sales per day is Poisson with mean 5, then generate 30 days of sales by Monte-Carlo method.* [Kanpur M.Sc. (Math.) 96]

Solution. Here the sales follow Poisson distribution with mean equal to 5. So we calculate the probabilities for demand from 0 to 12.

The probability for sales s is given by $P(X = s) = \frac{e^{-m} m^s}{s\,!}$, where $m = 5$ (given).

Let $x = e^{-5}$, then by taking logarithm, we have

$\log x = -5 \log e = -5 \times 0.4343 = -2.1715 = -3 + 0.8285 = \bar{3}.8285$ $\therefore$ $x = 0.006738$ (taking anti-log)

The cumulative probabilities for $s = 0, 1, 2, ..., 12$ are computed below :

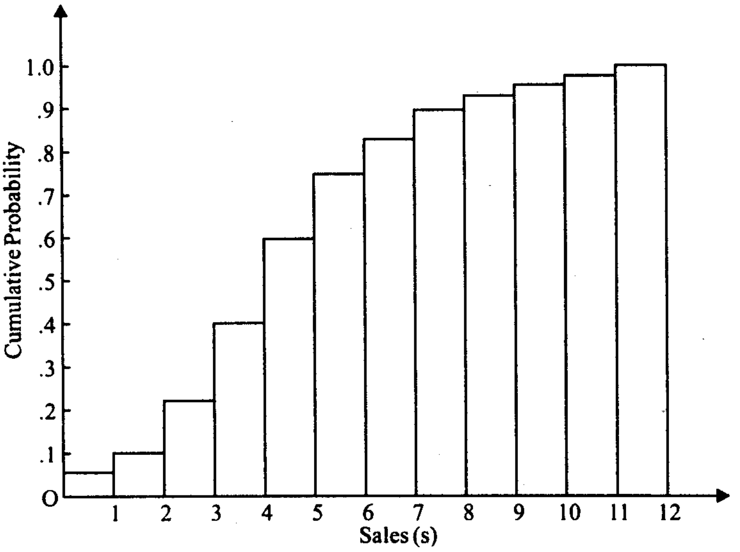


Fig. 17.2. Cumulative Distribution Graph.

Table 17.7.

Value of *s*	Cumulative Probability	Value of *s*	Cumulative Probability
0	.0067	7	.8666
1	,0404	8	.9319
2	.1247	9	.9682
3	.2650	10	.9763
4	.4405	11	.9845
5	.6160	12	1.0000
6	.7622		

We can draw a cumulative distribution graph between the sales *s* and the cumulative probablities.

Now, we take 30 two-digit random numbers from the random number tables and read the corresponding values of sales *s* from the graph. These values will give us the sales for 30 days. The values are tabulated below :

Table 17.8.

Random Number	Sales (*s*)	Random Number	Sales (*s*)	Random Number	Sales (*s*)
10	02	81	07	46	04
48	04	64	05	57	05
01	00	79	06	32	03
50	04	16	02	55	05
11	02	46	04	95	09
01	00	69	06	85	07
53	05	17	03	39	04
60	05	92	08	33	03
20	03	23	03	09	02
11	02	68	06	93	09

Q. 1. Explain how simulation can be applied in the case of invertory control where the demand is probabilistic and lead time is random. **[Indore 94]**

2. Write a detailed note on applications of simulation in inventory problem. **[Delhi 92]**

17.11-2. Application to Queueing Problems

In fact, some queueing problems cannot be solved explicitly by analytical methods. In such cases, the only possible method of solution is to simulate the experiment.

The method of simulation technique for queueing problems is illustrated below.

Example 9. *Records of 100 truck loads of finished jobs arriving in a department's check-out area show the following : Checking out takes 5 minutes and checker takes care of only one truck at a time. The data is summarized in the following table :*

Truck inter-arrival time (min.)	:	1	2	3	4	5	6	7	8	9	10
Frequency	:	1	4	7	17	31	23	7	5	3	2 (Total = 100)

As soon as the trucks are checked out, the truck drivers take them to the next departments. Using Monte-Carlo simulation, determine :

(a) *What is the average waiting time before service?* (b) *What is likely to be the longest wait ?*

Solution. From the given distribution of truck arriving times, we construct a cumulative probability distribution, as shown in the following table. This table enable us to select the range of random numbers for which we shall choose a prescribed value of time.

Table 17.9

Period (length = 1 min)	Frequency	Probability	Cumulative Prob.	Random Numbers
1	1	0.01	0.01	0
2	4	0.04	0.05	1-4
3	7	0.07	0.12	5-11
4	17	0.17	0.29	12-28
5	31	0.31	0.60	29-59
6	23	0.23	0.83	60-82
7	7	0.07	0.90	83-89
8	5	0.05	0.95	90-94
9	3	0.03	0.98	95-97
10	2	0.02	1.00	98-99

Now, the *Table* 17.9 becomes the basis for generating arrival and service times. A simulation work-sheet can be prepared in the following way (see *Table* 17.10)

First we select 20 two digit random numbers from the random number table. The first random number for the arrival time is 12. This number lies in the range (12-28), as shown in *Table* 17.9. Therefore, this random number indicates the arrival time at 10.04 *a.m.* assuming that the checking starts at 10.00 *a.m.* Similarly, we can work-out all the simulated arrivals and service times. Since the first truck arrives at 10.04 *a.m.*, the checker waits for 4 minutes. This is indicated in the last column, as checker's waiting time in *Table* 17.10. The checker takes 5 minutes and thus the service for first truck will end at 10.09 *a.m.* The next truck will arive at 10.10 *a.m.* which indicates that the checker waits for 1 minute. Whenever the truck has to wait because of the checker being busy in dealing with previous truck, the waiting time is listed in the last column of *Table* 17.10. The similar procedure can be adopted to prepare the entire work-sheet.

Table 17.10

S. No.	**Random Number**	**Inter-arrival time (min)**	**Arrival time**	**Service begins**	***Service ends***	**Waiting time for**	
						Checker	**Truck**
1	12	4	10.04	10.04	10.09	4	–
2	81	6	10.10	10.10	10.15	1	–
3	36	5	10.15	10.15	10.20	–	–
4	82	6	10.21	10.21	10.26	1	–
5	21	4	10.25	10.26	10.31	–	1
6	74	6	10.31	10.31	10.36	–	–
7	90	8	10.39	10.39	10.44	3	–
8	55	5	10.44	10.44	10.49	–	–
9	79	6	10.50	10.50	10.55	1	–
10	70	6	10.56	10.56	11.01	1	–
11	14	4	11.00	11.01	11.06	–	1
12	59	5	11.05	11.06	11.11	–	1
13	62	6	11.11	11.11	11.16	–	–
14	57	5	11.16	11.16	11.21	–	–
15	15	4	11.20	11.21	11.26	–	1
16	18	4	11.24	11.26	11.31	–	2
17	74	6	11.30	11.31	11.36	–	1
18	11	3	11.33	11.36	11.41	–	3
19	41	5	11.38	11.41	11.46	–	3
20	29	5	11.43	11.46	11.51	–	3

After completing the simulation, following information can be obtained which describe the behaviour of single server counter as given below :

(*i*) Average waiting time of trucks before service $= \dfrac{\text{Total waiting time}}{\text{Total number of arrivals}} = 16/20 = 0.8$ minutes.

(*ii*) Expected longest period of waiting = 3 minutes.

Example 10. *Arrivals at a service station have been found to follow Poisson process. The mean arrival rate is* $\lambda = 6$ *units per hour. Simulate five hours of arrivals at the station.*

Solution. The probability of k arrivals by Poisson distribution is given by

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k\,!}, \text{ where } \lambda = 6.$$

The cumulative probabilities for k or less arrivals can be easily determined from the tables of cumulative Poisson distribution for $\lambda = 6$.

The values may be put in the following tabular form.

Table 17.11.

Number of arrivals (k) :	0	1	2	3	4	5	6	7	8	9	10	11	12 or more
Probability of k or less :	.0025	.0174	.0620	.1512	.2851	.4457	.6063	.7440	.8472	.9161	,9574	.9799	1.0000

We can draw a cumulative distribution graph (similar to *Fig. 17.2.*) between the number of arrivals (k) and the cumulative probabilities.

We now take 25 two-digit random numbers from the random number tables and read the corrsponding values of arrivals k from the graph.

Table 17.12.

Arrival No.	*Random number*	*k*	*60 × 1/k minutes*	*Arrival time (min.)*
1	10	3	20.0	20.0
2	48	6	10.0	30.0
3	01	2	30.0	60.0
4	50	6	10.0	70.0
5	11	3	20.0	90.0
6	01	2	30.0	120.0
7	53	6	10.0	130.0
8	60	6	10.0	140.0
9	20	4	15.0	155.0
10	11	3	20.0	174.0
11	81	8	7.5	182.5
12	64	7	8.5	191.0
13	79	8	7.5	198.5
14	16	3	20.0	218.5
15	46	6	10.0	228.5
16	69	7	8.5	236.5
17	17	4	15.0	251.5
18	92	10	6.0	257.5
19	23	4	15.0	272.5
20	68	7	8.5	281.0
21	46	6	10.0	291.0
22	57	7	8.5	299.5*
23	32	5	12.0	–
24	55	6	10.0	–
25	95	11	5.5	–
				Five hours completed*

Example 11. *Hundred unemployed people were found to arrive at a one-person state-employment office to obtain their unemployment compensation cheque according to the following frequency distribution.*

Inter-arrival time (min.)	*Frequency*	*Service Time (min.)*	*Frequency*
2	10	2	10
3	20	3	20
4	40	4	40
5	20	5	20
6	10	6	10

The Govt. is interested in predicting the operating characteristics of this one-person state-exmployment office during a typical operating day from 10.00 a.m. to 11a.m. Use simulation to determine the average waiting time and total time in the system, and the maximum queue length.

Solution. Step 1. From the prescribed distribution of arrivals and service times, the random numbers can be assigned to the arrival times as tabulated below :

Table 17.13.

Inter-arrival time (min.)	*Frequency*	*Probability*	*Cumulative Probability*	*Random Number*
2	10	0.10	0.10	00-09
3	20	0.20	0.30	10-29
4	40	0.40	0.70	30-69
5	20	0.20	0.90	70-89
6	10	0.10	1.00	90-99

Step 2. Proceeding as in *Step 1*, the random number can be assigned to service times as shown in the following table.

Table 17.14.

Service time (min.)	Frequency	Probability	Cumulative Prob.	Random Number
2	10	0.10	0.10	00-09
3	20	0.20	0.30	10-29
4	40	0.40	0.70	30-69
5	20	0.20	0.90	70-89
6	10	0.10	1.00	90-99

Step 3. ***To develop the simulation work-sheet as shown in Table 17.15.***

We select a list of random numbers for arrival time is 17, corresponding to inter-arrival time of 3 minutes. This indicates that first person arrives 3 minutes later the service window opens. So the service can be started immediately at 10.03 *a.m.*

The first random number in column 6 of *Table 17.15* is 90, corresponding to a service time of 6 minutes. So the first person will leave the system at 10.0 *a.m.* But, the first person spent 6 minutes in the system, and there was no queue at the time of arrival.

Table 17.15.

S. No.	Random No.	Inter-arrival Time (min.)	Arrival Time	Service Starts	Random No.	Service		Waiting Time of		Time in System	Queue Length
						Time (mts.)	End (a.m.)	Server (mts.)	People (mts.)		
1	17	3	10:03	10:03	90	6	10:09	3	–	6	–
2	86	5	10:08	10:09	59	4	10:13	–	1	5	1
3	84	5	10:13	10:13	95	6	10:19	–	–	6	–
4	79	5	10:18	10:19	68	4	10:23	–	1	5	1
5	33	4	10:22	10:23	51	4	10:27	–	1	5	1
6	55	4	10:26	10:27	82	5	10:32	–	1	6	1
7	06	2	10:28	10:32	72	5	10:37	–	4	9	1
8	42	4	10:32	10:37	01	2	10:39	–	5	7	2
9	93	6	10:38	10:39	77	5	10:44	–	1	6	1
10	38	4	10:42	10:44	80	5	10:49	–	2	7	1
11	58	4	10:46	10:49	84	5	10:54	–	3	8	1
12	71	5	10:51	10:54	19	3	10:57	–	3	6	1
13	74	5	10:56	10:57	34	4	11:01	–	1	5	1

The second person is associated with the random number 86, which indicates that the arrival time is 5 minutes. Since the service of first person ended at 10.09 *a.m.*, the next person has to wait for 1 minute before the starting of his service. The next service time is associated with random number 59 indicating that the service will last for 4 minutes (*Table 17.14*). Since the server is busy, the number of people waiting in the queue is shown in the column '*queue-length*'. The simulation continues in the like manner until the closing hours.

Step 4. ***To Compute the required information from the simulation-sheet.***

(*i*) Average waiting time of people before service $= \dfrac{\textit{waiting time of the pepole}}{\textit{total number of arrivals}} = {}^{23}/_{13} = 1.77$ minutes

(*ii*) Average queue length $= \dfrac{\textit{total number of people in the queue}}{\textit{total number of arrivals}} = {}^{12}/_{13} = 1$ (nearly)

(*iii*) Average time the person spends in the system

= Average service time + Average waiting time before the service

= 4.46 + 1.77 = 6.23 minutes.

Q. 1. Write a note on the use of Monte-Carlo methods in waiting line problems.

2. Show how simulation technique can be used in solving : (i) The problem of model building (ii) Waiting line problems.
3. How can simulation be applied for (i) Queueing Problems (ii) Inventory Problems.
4. Explain simulation and give its applications to queueing theory.

17.11-3. Application to Capital-budgeting

An important decision of financial analysis is to select the optimum alternative among various capital investment policies and evaluation of risk involved with the specific decision. The main purpose of evaluating the risk is to determine the effect of various factors (*e.g. selling price, market growth rate, market size*, etc.) on financial parameters. The samples from the probability distributions of the factors involved can be drawn and analysed to determine the rate of return on investment.

While evaluating the rising investments, ***Prof. David B. Hertz*** (1964) proposed the use of a simulation model to obtain the expected return for an investment proposal. The method involves the following steps :

Step 1. ***To develop the probability distribution of uncertain factors.***

The probability distributions are developed on the basis of assessment of the probable outcomes. The following factors are taken into consideration for evaluating an investment proposal.

(*i*) *Market size* (*ii*) *Selling price* (*iii*) *Market growth rate*
(*iv*) *Share of market* (*v*) *Investment required* (*vi*) *Residual value of investment*
(*vii*) *Operating costs* (*viii*) *Fixed costs* (*ix*) *Useful life of facilities.*

Step 2. ***To generate a set of random numbers***

A set of random numbers is generated and respective probability distributions are assigned to each factor in the system for calculating the expected values of these factors. Then the values of all the factors are combined to determine the rate of return for the combination.

Step 3. ***To simulate the process.***

Above process is repeated several times to simulate a clear portrayal of investment risk.

The following example will make the process clear.

Illustration. To understand, let us consider the following case.

A medium sized industrial chemical producer is considering a $ 10 million extension to its processing plant. The estimated service life of the facility is 10 years, the engineers expect to be able to utilize 2,50,000 tonnes of processed material worth $ 510 per tonne at an average processing cost of $ 435 per tonne. Is this investment a good bet?

Step 1. Estimate the range of values for each of the factors like range of selling price, sales growth rates and so on and within that range, the liklihood of occurrence of each value, *i.e.* develop the probability distribution of uncertain factors in the system.

For example, estimates for factor of selling price for the finished product can be determined by asking following questions :

(*i*) Given that $ 510 is the expected sales price, what is the probability that the price will exceed $550 ?

(*ii*) Is there any chance that the price will exceed $ 650 ?

(*iii*) How likely is it that the price will drop below $ 475 ?

Management must ask similar questions for each of the other factors, until they can construct a curve for each factor and this is not that difficult to perform. Often information on the degree of variation of factor is readily available.

Step 2. The next step in the proposed approach is to select at random (from the distribution of values for each factor) one particular value and associate each one of them with the raspective probability distribution to determine the expected value for each factor in the model. Then combine the values of all the factors and compute the rate of return (or present value) from that combination.

For example, the lowest in the range of prices might be combined with the highest in the range of growth rate and other factors. The fact that the factors are dependent should be taken into consideration.

Step 3. Perform this over and over again to define and evaluate the odds of the occurrence of each possible rate of return. Since there are literally millions of possible combinations of values, we need to test the likelihood that various specific returns on the investment will occur. The result will be a listing of the rates of return we might achieve, ranging from a loss (if the factors go against us) to whatever maximum gain is possible with the estimates that have been made. For each of these rates, the chances that it may occur are determined. The average expectation is the average of the values of all outcomes weighted by the chances of each occurring.

The variability of outcome values from the average is also determined. This will help the management to select investments having the same return but with lower variability. When the expected return and variability of each of a series of investments have been determined, the same procedure may be used to study the effectiveness of various combination of them in meeting management objectives.

A computer can be used to carry out the trials for the simulation process so as to obtain more accurate results. The process is repeated a number of times, each time we obtain a combination of values for all the factors (i) to (ix) and the corresponding return on investment. When sufficient number of trials are carried out, a graph can be plotted for the rates of return and also obtain a frequency distribution. From this frequency distribution, the expected return or risk can be evaluated. In other words, we can determine the probability that an investment will provide a return greater than or less than a certain amount. By comparing the probability distribution of the rates of return, the management is in a position to evaluate the respective merits of different investments involving risks.

Q. Discuss the steps in the HERTZ simulation model, with the help of an illustration. **[C.A. (Nov.) 93, 89]**

Example 12. *An Investment corporation wants to study the investment projects based on three factors : market demand in units, price per unit* **minus** *cost per unit, and the investment required. These factors are felt to be independent of each other. In analysing a new consumer product, the corporation estimates the following probability distrbutions :*

Table 17.16.

Annual Demand		(Price–Cost) per unit		Investment Required	
Units	Probability	Rs.	Probability	Rs.	Probability
20,000	0.05	3.00	0.10	17,50,000	0.25
25,000	0.10	5.00	0.20	20,00,000	0.50
30,000	0.20	7.00	0.40	25,00,000	0.25
35,000	0.30	9.00	0.20		
40,000	0.20	10.00	0.10		
45,000	0.10				
50,000	0.-05				

Using simulation process, repeat the trial 10 times, compute the return on investment for each trial taking these three factors into account. Approximately, what is the most likely return ?

Solution. The yearly return can be determined by the formula :

$$\textbf{Return } (R) = \frac{\textbf{(Price – Cost)} \times \textbf{Number of units demanded}}{\textbf{Investment}}$$

To determine a cumulative probability distribution corrsponding to each of the three factors, we assign an appropriate set of random numbers representing each of the three factors as shown in *Tables* 17.17, 17.18, and 17.19.

Table 17.17

Annual Demand	Probability	Cumulative Prob.	Random Number
20,000	0.05	0.05	00-04
25,000	0.10	0.15	05-14
30,000	0.20	0.35	15-34
35,000	0.30	0.65	35-64
40,000	0.20	0.85	65-84
45,000	0.10	0.95	85-94
50,000	0.05	1.00	95-99

Table 17.18.

(Price–Cost) per unit	Probability	Cumulative Prob.	Random Number
3.00	0.10	0.10	00-09
5.00	0.20	0.30	10-29
7.00	0.40	0.70	30-69
9.00	0.20	0.90	70-89
10.00	0.10	1.00	90-99

Table 17.19.

Investment Required	Probability	Cumulative Prob.	Random Number
17,50,000	0.25	0.25	00-24
20,00,000	0.50	0.75	25-74
25,00,000	0.25	1.00	75-99

We prepare the simulation work-sheet for 10 trials. The simulated return (R) is also calculated by using the formula for R. The results of simulation are shown in *Table 17.20.*

Table 17.20.

Trials	Random Number of Demand	Simulated Demand ('000)	Random Number for Profit (Price-Cost) per unit	Simulated Profit	Random Number for Investment	Simulated Investment ('000)	Simulated Return (%) $= \frac{Demand \times Profit\ per\ unit}{investment} \times 100$
1	28	30	19	5.00	18	1750	8.57
2	57	35	07	3.00	67	2000	5.25
3	60	35	90	10.00	16	1750	20.00
4	17	30	02	3.00	71	2000	4.50
5	64	35	57	7.00	43	2000	12.25
6	20	30	28	5.00	68	2000	7.50
7	27	30	29	5.00	47	2000	7.50
8	58	35	83	9.00	24	1750	18.00
9	61	35	58	7.00	19	1750	14.00
10	30	30	41	7.00	97	2500	8.40

Result. *Table 17.20* shows, the highest likely return is 20% which is corresponding to the annual demand of 35,000 units resulting a profit of Rs. 10 per unit and the required investment will be 17,50,000.

17.11-4. Application to Financial Planning

Application of simulation to the problems of *financial planning* can be easily understood with the help of following general problem :

Example 13. *Modern business corporation faces major financial decision making problems in varying situations and at different stages of its development. Preparation of the company's budget for the management decision is one of the most complex problem with which the planning department is faced with. Following are critical factors to be decided for the formation of a budget, by the manager of a company during the decision making process :*

(i) Amount of money to be borrowed, (ii) Amount of existing debt to be, (iii) Amount of dividend to be paid.

(iv) Amount of money to be allocated to production, (v) Amount of money to be allocated for research.

(vi) Amount of money to be allocated for marketing.

(vii) Amount of money to be allocated to capital investment in production and research facilities, and so on.

Solution. Formulation of Model. The manager realises that expenditure today must be motivated by the objective to improve future as well as current returns. Therefore, he must take a long term view-point in terms of say a 15-year time period. He wants to make his decisions in such a way that the 15 years profits are high as well as the future prospects of the firm at the end of 15 years are good. Consequently, his goals are :

1. To maximize the discounted streams of profits during time horizon.
2. To maximize the firm's market share at the end of the time horizon.
3. To maximize the firms physical capital in the form of production and research facilities, *i.e.* to maximize the capital existing at the end of the time horizon.

The manager realises that improving any one of these objectives can leave a smaller budget for other two objectives. Thus given a budget constraint, he cannot liberally try to maximize all three simultaneously.

First, he must choose one of the goals as an objective and try to attain his other goals through the use of constraint. Suppose his problem is :

To maximize total discounted 15 years profits, subject to the constraints :

Market share at the end of 15 *years* $\geq M$ (where M is the existing market share)

Physical capital at the end of 15 *years* $\geq C$ (where C is existing capital).

The parameters M and C may be set at various levels to determine their effect on the optimal objective value. He may then finally decide on the values to specify for his targets as policy objectives.

Solution Procedure. Clearly, above model would be very difficult to explicitly construct and solve. Because, there are many variables, many interactions and uncertainties and many of the required functional forms could only be estimated. For example, revenue depends on the level of production, but the amount one decide to produce depends on our forecast for future demand which can be obtained from a forecasting model. This problem can be easily solved by simulating the manager's problem on a period by period basis. The output of one year's activities forms part of the input for the next year. In any one year (say, t) it receives the following inputs :

1. (a) The set of external parameters to the model such as current interest rate, GNP etc.
   (b) The set of managerial decisions.
2. The simulated output from the previous year.

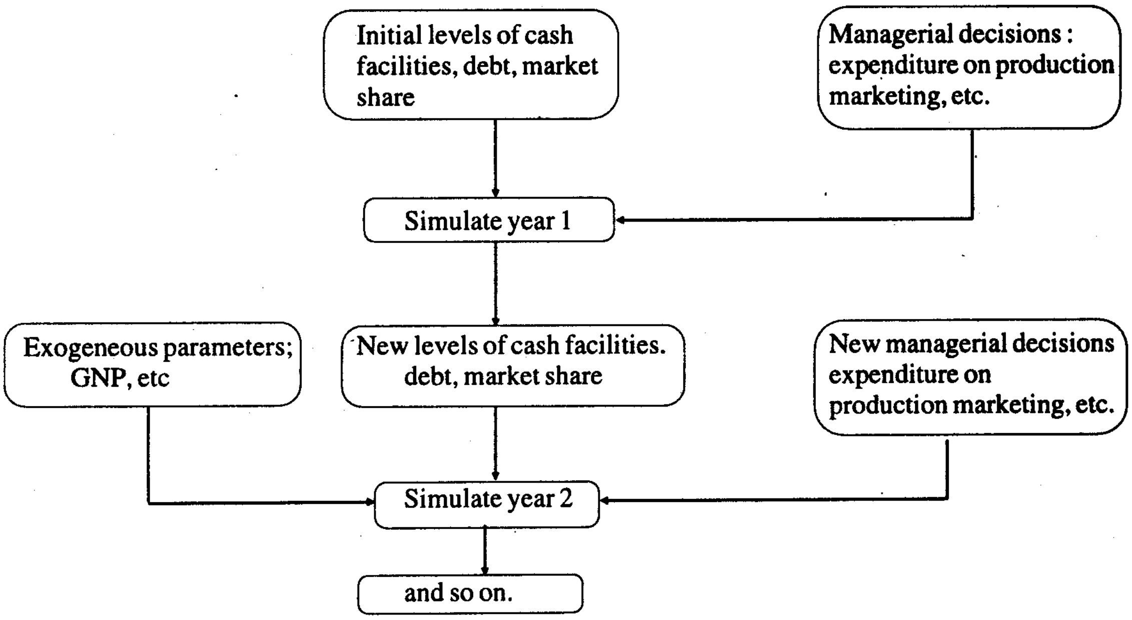


Fig. 17.3 The simulation process of financial planning.

Since 0 year does not exist, it is obvious that an initial set of 'outputs' must be provided to do the simulation in year 1. The simulator itself contains models (a simulation model is often a collection of submodels) that reproduce and update pertinent aspects. These models use the inputs to make various endogenous decisions like price and the quantity produced.

The overall model then simulates the behaviour of the system under the conditions that would exist if these inputs and endogenous decisions had actually occurred. It does so using random numbers to select 'actual demand' from a statistical distribution and then by determining what would happen if his demand had occurred. The appropriate distribution of demand is itself determined by some combination of external inputs (exogenous parameters and management decisions) and endogenous decisions. For example, the distribution of demand might be determined by the general state of the economy (an exogenous parameter) the level of marketing activity (a managerial decision) and the price (an endogenous decision). Given the price, quantity produced and the actual demand the simulator can calculate the gross revenue and profits. The quantities determine new financial position, which is part of the internally generated input to the next period of activity. In this next period, new values for the exogenous parameters are input. Financial results from the previous time period, augments possibly by a decision to borrow at the beginning of this new period are now used to determine the available funds for a new set of budget allocations (see Fig. 17.3). The model continues to operate in this manner over a sequence of time periods. Thus, with a given sequence of exogenous parameter are managerial decisions, the unfolds the future, and discounted future profits over any given time horizon can be obtained. In-this manner, the model can be used as a tool to explore some of the interactions between various managerial policies and exogenous conditions.

Q. Describe the application of simulation to the probolems of Financial Planning with an illustration. **[C.A. (May) 90]**

Example 14. *The director of finance for a farm cooperative is concerned about the yields per acre she can expect from this year's corn crop. The probability distribution of the yields for the current weather conditions is given below :*

Yield in kg. per acre	:	120	140	160	180
Probability	:	0.18	0.26	0.44	0.12

She would like to see a simulation of the yield she might expect over the next 10 years for weather conditions similar to those she is new experiencing.

(*i*) *Simulate the average yield she might expect per acre using the following random numbers :*

20, 72, 34, 54, 30, 22, 48, 74, 76, 02.

She is also interested in the effect of market price fluctuations on the cooperative's farm revenue she makes this estimate of per kg. prices for corn.

Price per kg. (Rs.)	:	2.00	2.10	2.30	2.30	2.40	2.50
Probabilities	:	0.05	0.15	0.30	0.25	0.15	0.10

(*ii*) *Simulate the price she might expect to observe over the next 10 years using the following random numbers :*

82, 95, 18, 96, 20, 84, 56, 11, 52, 03. **[C.A. (May) 91]**

Solution. If the numbers 0–99 are allocated in proportion to the probabilities associated with each category of yield per acre, then various kinds of yields can be sampled using random number table.

Table 17.21.

Yield in kg. per acre	:	120	140	160	180
Probability	:	0.18	0.25	9.44	0.12
Cumulative probability	:	0.18	0.44	0.88	1.00
Random numbers assigned	:	00–17	19–43	44–87	88–99

(*i*) Let us now simulate the yield per acre for the next 10 years based on the given 10 random numbers.

Table 17.22

Year	:	1	2	3	4	5	6	7	8	9	10
Random number	:	20	72	34	54	30	22	48	74	76	02
Simulated yield	:	140	160	140	160	140	140	160	160	160	120 (Total 1480)

The average yield = $1480/10$ = 148 kg/acre

(*ii*) Let us now simulate the price she might expect in the next 10 years based on the random numbers given.

Table 17.23

Price/kg	:	2.00	2.10	2.20	2.30	2.40	2.50
Probability	:	0.05	0.15	0.30	0.25	0.15	0.10
Cumulative Prob.	:	0.05	0.20	0.50	0.75	0.90	1.00
Random no. assigned	:	00–04	05–19	20–49	50–74	75–89	90–99

These simulated prices are developed using random numbers given for next 10 years.

Table 17.24

(1)	Year	:	1	2	3	4	5	6	7	8	9	10
(2)	Random no.	:	82	95	18	96	20	84	56	11	52	03
(3)	Price (Rs,.)	:	2.40	2.50	2.10	2.50	2.20	2.40	2.30	2.10	2.30	2.00
(4)	Simulated yield	:	140	160	140	160	140	140	160	160	160	120
(5)	Revenue/acre (Rs,)	:	336	400	294	400	308	336	368	336	368	240

(iii) Average revenue per acre = Rs. (3386/10) = Rs. 338.60.

Example 15. *The occurrence of rain on a city an a day is dependent upon whether or not it rained on the previous day. If it rained on the previous day, the rain distribution is given by :*

Event	:	No rain	1 cm. rain	2 cm. rain	3 cm. rain	4 cm. rain	5 cm. rain
Probability	:	0.50	0.25	0.15	0.05	0.03	0.02

If it did not rain the previous day, the rain distribution is given by :

Event	:	No rain	1 cm. rain	2 cm. rain	3 cm. rain
Probability	:	0.75	0.15	0.06	0.04

Simulate the city's weather for 10 *days and determine by simulation the total days without rain as well as the total rainfall during the period. Use the following random numbers.*

67 63 39 55 29 79 70 06 78 76

for simulation. Assume that for the first day of the simulation it had not rained the day before.

[C.A. (Nov.) 93]

Solution The number 00–99 are allocated in proportion to the probabilities associated with each event. If it rained on the previous day, the rain distribution and the random number allocation are given below :

Table 17.25. Rain on Previous Day

Event (rain)	:	No	1 cm.	2 cm.	3 cm.	4 cm.	5 cm.
Probability	:	0.50	0.25	0.15	0.05	0.03	0.02
Cum prob.	:	0.50	0.75	0.90	0.95	0.98	1.00
Random nos. assigned	:	00–49	50–74	75–89	90–94	95–97	98–99

Similarly if it did not rain the previous day, the necessary distribution and the random number allocation is given below :

Table 17.26. No Rain on Previous Day

Event (rain)	:	No	1 cm.	2 cm.	3 cm.
Probability	:	0.75	0.15	0.06	0.04
Cum prob.	:	0.75	0.90	0.96	1.00
Random Nos. Assigned	:	00–74	75–89	90–95	96–99

Let us now simulate the rain fall for 10 days using the given random numbers. For the first day it is assumed that it had not rained the day before :

Table 17.27. No rain on previous day

Day	:	1	2	3	4	5	6	7	8	9	10
Random nos.	:	67	63	39	55	29	78	70	06	78	76
Event (rain)	:	No	No	No	No	No	1 Cm.	1 Cm.	No	1 Cm.	2 Cm
(From Table No.)	:	4.26	4.26	4.26	4.26	4.26	4.26	4.25	4.25	4.25	2.45

Hence, during the simulated period, on 6 days out of 10 days. The total rain fall during the period was 5 cm.

Example 16. *The output of a production line is checked by an inspector for one or more of three different types of defects, called defects A, B and C. If defect A occurs, the item is scrapped. If defect B or C occurs, the item must be reworked. The time required to rework a B defect is 15 minutes and the time required to rework a C defect is 30 minutes. The probabilities of an A, B and C defects are 0.15, 0.20, 0.10 respectively. For ten items coming off the assembly line, determine the number of items without any defects, the number scrapped and the total minutes of rework time. Use the following random numbers :*

Random nos. for defect A	:	48	55	91	40	93	01	83	63	47	52
Random nos. for defect B	:	47	36	57	04	79	55	10	13	57	09
Random nos. for defect C	:	82	95	18	96	20	84	56	11	52	03

[C.A. (May) 94]

Solution. The probabilities of occurrence of A, B and C defects are 0.15, 0.20 and 0.10 respectively. So the numbers 00–99 the allocated item in proportion to the probabilities associated with each of the three defects :

Defect A		Defect B		Defect C	
Exists (Yes/No)	Random Nos.	Exists (Yes/No)	Random Nos.	Exists (Yes/No)	Random Nos.
Yes	00–14	Yes	00–19	Yes	00–09
No	15–99	No	20–99	No	10–99

Let us now simulate the output of the assembly line for 10 items using the given random numbers in order to determine the number of items without any defect, the number of items scrapped and the total minutes of rework time required :

Item No.	Random Numbers for Defects			Whether any Defect Exists?	Rework Time (in minutes)	Remarks
	A	B	C			
1	48	47	82	None	—	—
2	55	36	95	None	—	—
3	91	57	18	None	—	—
4	40	04	96	B	15	—
5	93	79	20	None	—	—
6	01	55	84	A	—	Scrap
7	83	10	56	B	15	—
8	63	13	11	B	15	—
9	47	57	52	None	—	—
10	52	09	03	B, C	15 + 30 = 45	—

During the simulated period, 5 items out of 10 had no defects, one item was scrapped and 90 minutes of total rework time was required by 3 items.

Example 17. *The management of ABC company is considering the question of marketing a new product. The final cost required in the project is Rs. 4,000. Three factors are uncertain viz the selling price variable cost and the annual sales volume. The product has a life of only one year. The management has the data on these three factors as under :*

Selling Price	*Prob.*	*Variable Cost (Rs.)*	*Prob.*	*Sales Volume (Units)*	*Prob.*
3	0.2	1	0.3	2,000	0.3
4	0.5	2	0.6	3,000	0.3
5	0.3	3	0.1	5,000	0.4

Consider the following sequence of random numbers :

81, 32, 60, 04, 46, 31, 67, 25, 24, 10, 40, 02, 39, 68, 08, 59, 66, 90, 12, 64, 79, 31,
86, 68, 82, 89, 25, 11, 98, 16.

Using the sequence (first 3 random numbers for the first trial, etc.), simulate the average profit for the above project on the basis of 10 trials. **[C.A. (Nov.) 94]**

Solution. First we allocate the random numbers 00–99 in proportion to the probabilities associated with each of the three variables as given under :

	Selling price			Variable cost (Rs.)			Sales volume (Units)		
	Rs. 3	Rs. 4	Rs. 5	Rs. 1	Rs. 2	Rs. 3	2,000	3,000	5,000
Probabilities :	0.2	0.5	0.3	0.3	0.6	0.1	0.3	0.3	0.4
Cum. prob. :	0.2	0.7	1.00	0.3	0.9	1.00	0.3	0.6	1.00
Random nos. assigned :	00–19	20–69	70–99	00–29	30–89	90–99	00–29	30–59	60–99

Let us now simulate the output of ten trials using the given random numbers in order to find the average profit for the project.

S. No.	Random No.	Selling Price (Rs.)	Random No.	Variable cost (Rs.)	Random No.	Sales Volume ('000 units)
1	81	5	32	2	60	5
2	04	3	46	2	31	3
3	67	4	25	1	24	2
4	10	3	40	2	02	2
5	39	4	68	2	08	2
6	59	4	66	2	90	5
7	12	3	64	2	79	5
8	31	4	86	2	68	5
9	82	5	89	2	25	2
10	11	3	98	3	16	2

Profit = (Selling price—Variable cost) × Sales volume—Fixed cost

Simulated profit in 10 trials would be as follows :

S. No.	Selling Price–Variable Cost × Sales volume – Fixed Cost	Profit (Rs.)
1	(Rs. 5 – Rs.2) × 5000 units – Rs. 4,000	11,000
2	(Rs. 3 – Rs. 2) × 3,000 units – Rs. 4000	–1,000
3	(Re 4 – Re 1) × 2,000 units – Rs. 4000	2,000
4	(Rs. 3 – Rs 2) × 2,000 units – Rs. 4,000	–2,000
5	(Rs. 4 – Rs.2) × 2,000 units – Rs. 4,000	0
6	(Rs. 4 – Rs. 2) × 5,000 units – Rs. 4,000	6,000
7	(Rs. 3 – Rs. 2) × 5,000 units – Rs. 4,000	1,000
8	(Rs. 4 – Rs. 2) × 5,000 units – Rs. 4,000	6,000
9	(Rs. 5 – Rs. 2) × 2,000 units – Rs. 4,000	2,000
10	(Rs. 3 – Rs. 3) × 2,000 units – Rs. 4000	–4,000

Therefore, average profit per trial = $\frac{\text{Rs. } 21{,}000}{10}$ = Rs. 2,100.

Example 18. *A bakery keeps stock of popular brand of bread. Previous experience indicates the daily demand as given below* ;

Daily demand :	0	10	20	30	40	50
Probability :	0·01	0·20	0·15	0·50	0·12	0·02

Consider the following sequence of random numbers :

48, 78, 19, 51, 56, 77, 15, 14, 68, 8

Using above sequence, simulate the demand for the next 10 days.

(i) Find out the stock situation if the owner of the bakery decides to make 30 breads every day.

(ii) Estimate the daily average demand for the bread on the basis of simulated data.

[AIMS (MBA) 2002; C.A., Nov. 98]

Solution. According to the given distribution of demand, the random number coding for various demand levels is shown in table 1 below :

Table 17.28 : Random Number Coding

Demand	Probability	Cumulative probability	Random number interval
0	0·01	0·01	00
10	0·20	0·01 + 0·20 = 0·21	01–20
20	0·15	0·21 + 0·15 = 0·36	21–35
30	0·50	0·36 + 0·50 = 0·86	36–85
40	0·12	0·86 + 0·12 = 0·98	86–97
50	0·02	0·98 + 0·02 = 1·00	98–99

The simultated demand for the breads for the next 10 days is given in table 2 below :

Table 17.29 : Determination Of Demand Levels

Day	Random number	Simulated demand	Stock situation, if 30 breads are produced daily
1	48	30	—
2	78	30	—
3	19	10	20
4	51	30	20
5	56	30	20
6	77	30	20
7	15	10	40
8	14	10	60
9	68	30	60
10	09	10	80

Average demand, on the basis of simulated data = $\frac{220}{10}$ = 22 breads/day.

Example 19. *The automobile company manufactures around 150 scooters. The daily production varies from 146 to 154 depending upon the availability of raw materials and other working conditions :*

Production per day	Probability
146	0·04
147	0·09
148	0·12
149	0·14
150	0·11
151	0·10
152	0·20
153	0·12
154	0·08

The finished scooters are transported in a specially arranged lorry accommodating 150 scooters. Using following random numbers :

80, 81, 76, 75, 64, 43, 18, 26, 10, 12, 65, 68, 69, 61, 57, simulate the process to find out :

(i) What will be the average number of scooters waiting in the factory ?

(ii) What will be the average number of empty space on the lorry ? **[C.A., May 98]**

Solution. The random numbers are restablished as in table 1 below :

Table 17.30 : Random Number Coding

Production per day	Probability	Cumulative probability	Random number assigned
146	0·04	0·04	00–03
147	0·09	0·13	04–12
148	0·12	0·25	13–24
149	0·14	0·39	25–38
150	0·11	0·50	39–49
151	0·10	0·60	50–59
152	0·20	0·80	60–79
153	0·12	0·92	80–91
154	0·08	1·00	92–99

Based on the 15 random numbers given, we simulate the production per day in the table below :

Table 17.31 : Simulation Sheet

S. No.	Random number	Production per day	No. of scooters waiting	No. of empty spaces in the lorry
1	80	153	3	
2	81	153	3	
3	76	152	2	
4	75	152	2	
5	64	152	2	
6	43	150	0	0
7	18	148		2
8	26	149		1
9	10	147		3
10	12	147		3
11	65	152	2	
12	68	152	2	
13	69	152	2	
14	61	152	2	
15	57	151	1	
Total			21	9

(i) Average number of scooters waiting $= \frac{21}{15} = 1{\cdot}4$ per day

(ii) Average number of empty spaces $= \frac{9}{15} = 0{\cdot}6$ per day.

Example 20. (i) *A businessman is considering taking over a certain new business. Based on past information and his own knowledge of the business, he works out the probability distributions of the daily costs and sales revenues, as given here :*

Cost (in Rs.)	Probability	Sales	Probability
8,500	0·10	9,500	0·10
9,000	0·10	10,000	0·10
9,500	0·40	10,500	0·20
10,000	0·20	11,000	0·40
10,500	0·20	11,500	0·15
		12,000	0·05

Use the following sequences of random numbers to be used for estimating costs and revenues. Obtain the probability distribution of the daily net revenue.

Sequence 1 : 81, 83, 27, 81, 35, 91, 72, 90, 62, 28, 26, 25, 91, 62, 82, 02, 12, 38, 10, 18.

Sequence 2 ; 38, 71, 28, 70, 82, 18, 71, 91, 58, 48, 38, 71, 92, 02, 91, 73, 17, 09, 04

(ii) Repeat the analysis in (i) by using the following random number streams :

Sequence 1 : 21, 64, 47, 17, 46, 42, 45, 67, 88, 27, 79, 41, 30, 93, 22, 37, 58, 04, 29, 09.

Sequence 2 : 24, 58, 99, 52, 30, 42, 12, 67, 31, 42, 81, 63, 75, 65, 27, 42, 41, 98, 16.

[Gujarat (M.B.A.) Nov. 98]

Solution. *Step 1.* Obtain random number intervals :

Cost	Prob.	Cumprob.	Random number	Revenue	Prob.	Cumprob.	Random number
8,500	0·10	0·10	00–09	9,500	0·10	0·10	00–09
9,000	0·10	0·20	10–19	10,000	0·10	0·20	10–19
9,500	0·40	0·60	20–59	10,500	0·20	0·40	20–39
10,000	0·20	0·80	60–79	11,000	0·40	0·80	40–79
10,500	0·20	1·00	80–99	11,500	0·15	0·95	80–94
				12,000	0·05	1·00	95–99

Step 2 : Simulate cost and revenue data using given random numbers.

Part (i)					Part (ii)				
R. No.	Cost (in '000 Rs.)	R. No.	Revenue (in '000 Rs.)	Net revenue	R. No.	Cost (in '000 Rs.)	R. No.	Revenue (in '000 Rs.)	Net revenue
81	10·5	38	10·5	0	21	9·5	24	10·5	1
83	10·5	71	11·0	0·5	64	10·0	58	11·0	1
27	9·5	37	10·5	1·0	47	9·5	99	12·0	2·5
81	10·5	28	10·5	0	17	9·0	85	11·5	2·5
35	9·5	70	11·0	1·5	46	9·5	52	11·0	1·5
91	10·5	82	11·5	1·0	42	9·5	30	10·5	1·0
72	10·0	18	10·0	0	45	9·5	42	11·0	1·5
90	10·5	71	11·0	0·5	67	10·0	12	10·0	0
62	10·0	91	11·5	1·5	88	10·5	67	11·0	0·5
28	9·5	58	11·0	1·5	27	9·5	31	10·5	1·0
26	9·5	48	11·0	1·5	79	10·0	42	11·0	1·0
25	9·5	38	10·5	1·0	41	9·5	81	11·5	2·0
91	10·5	71	11·0	0·5	30	9·5	63	11·0	1·5
62	10·0	93	11·5	1·5	93	10·5	75	11·0	0·5
82	10·5	02	9·5	− 1·0	22	9·5	65	11·0	1·5
02	8·5	91	11·5	3·0	37	9·5	27	10·5	1·0
12	9·0	73	11·0	2·0	58	9·5	42	11·0	1·5
38	9·5	17	10·0	0·5	04	8·5	41	11·0	2·5
10	9·0	09	9·5	0·5	29	9·5	98	12·0	2·5
18	9·0	04	9·5	0·5	09	8·5	16	10·0	1·5

Step 3 : Prepare frequency distribution of net revenue and obtain probabilities of expressing frequencies in relative form.

	Part (i)		Part (ii)	
Net revenue	Frequency	Probability	Frequency	Probability
− 1,000	1	0·05	0	0·00
− 500	0	0·00	0	0·00
0	3	0·15	1	0·05
500	6	0·30	2	0·10
1,000	3	0·15	6	0·30
1,500	5	0·25	6	0·30
2,000	1	0·05	1	0·05
2,500	0	0·00	4	0·20
3,000	1	0·05	0	0·00

17.12. ADVANTAGES AND DISADVANTAGES

Advantages. In comparison to the mathematical programming and standard statistical analysis, the simulation technique has many advantages over these techniques. We summarize below a few important ones of them :

1. Simulation models are comparatively flexible and can be modified to adjust the variation in the environments of real situations.
2. The simulation is an easier technique to use than mathematical models and thus, considered quite superior to the mathematical analysis.
3. Simulation technique has the advantage of being relatively free from complicated mathematics and thus, can be easily understood by the operating staff and also by non-technical managers.

4. Simulation offers the solution by allowing experimentation with a model of the system without interfering with the real system. Thus, the simulation is often a bypass for complex mathematical analysis.
5. Computer simulation can compress the performance of a system over several years and involving large calculations into a few minutes of computer running time.
6. By simulation technique, thc management can foresee the difficulties and bottlenecks that may arise due to addition of new machines, equipment or process. Thus, this technique eliminates the need of costly trial and error methods of trying out the new concept on real methods and equipment.
7. It is always advantageous to train the people on simulated models before putting into their hands the costly real system. Simulated exercises have been developed to make the trainee expert and experienced. On account of his personal attachment, the trainee gains sufficient confidence, and moreover becomes familiar with data processing on electronic computer.

Disadvantages. Disadvantages of using simulation technique are indicated below.

1. Simulation is not precise. It is not an optimization process and does not yield an answer but merely provides a set of the system's responses to different operating conditions. In many cases, this lack of precision is difficult to measure.
2. Not all situations can be evaluated using simulation. Only the situations involving uncertainties can be tackled by simulation. Because, without a random component, all simulated experiments would provide the same answer.
3. A good simulation model may be too expensive. Often it takes years to develop an usable corporate planning model.
4. Simulation generates a way of evaluating solutions but does not generate the solution techniques. Users must still generate the solution approaches they want to test.

Q. 1. Discuss in detail the use of Monte-Carlo method in problems encountered in :
(i) Waiting line, and (ii) Storage, giving suitable illustration and pointing out the advantage of the method.
2. Describe the advantages and disadvantages of simulation models ?
3. What are the advantages and disadvantages of simulation models ? **[Meerut (M.com) 2005]**
4. Discuss simulation techniques with suitable examples. What are their advantages and disadvantages ? **[Meerut (Stat.) 95]**
5. Write short note on the Advantages of simulation **[C.A. (Nov.) 96]**

17.13. SIMULATION LANGUAGES

So far, we have discussed the material which provides a background for the study of the systems that need simulation. Now the next important thing is the formulation of the computer mode deciding how many variables can be included in the model. A simulation computer program can be written in one of the conventional computer languages such as : FORTRAN, COBOL, ALGOL or PL/1 without restoring to a special purpose simulation language. However, these languages require extensive programming experience. It can be observed that even in a simple queueing problem, various complicated details are involved in the simulation model. However, the development of special simulation languages has considerably reduced the program, preparation time and cost with features specially designed for different types of models and systems.

The special purpose simulation languages which have been developed so far are : GPSS, SIMSCRIPT, GASP, SIMPAC , DYNAMO, SIMULATE, CSL. The aim of these languages is to speed up the conversion of a simulation model to computer program. The several simulation languages are made so that they can be applied to different types of problems in order to make their simulation procedure more easy and automatic for the user.

The most popular among these languages are GPSS, SIMSCRIPT and DYNAMO. These three languages give automatic instructions for time keeping and other common simulation operations. The GPSS (General Purpose Systems Simulator) can be used to a wider class of systems in order to maintain a fixed set of procedures for performing the simulation automatically. There are four entities that constitute the foundations of GPSS : (*i*) transactions (*ii*) facilities (*iii*) storage, and (*iv*) blocks.

The basis of SIMSCRIPT language is a description of the system in terms of concepts of entity, attribute, set and event, etc. This language is more flexible than GPSS, but requires a knowledge of FORTRAN. Both of these languages are well-suited to *'queueing type'* problems.

The DYNAMO language is similar to SIMSCRIPT except that it simulates continuous type models instead of models involving individual events. This language is not particularly suitable for queueing type problems. Thus it seems to be more applicable to "Macro" simulations rather than "Micro" simulation which involve small-scale models of 'pieces' of a firm, such as *queueing, production scheduling* and *inventory*, etc.

17.14. SCOPE OF SIMULATION TECHNIQUE

Simulation is an approach rather than an application of a specific technique. Thus there is a wide range of applications of computer based simulation models. The major use of computer based *Monte-Carlo simulation* has been to solve the complex queueing problems. The GPSS program has been developed as a queueing simulation tool.

A large number of *job-shop* simulation programs have been developed which involve deterministic times for the individual operations of prescribed order. Because of different processing times for similar type operations and different order operation sequences, it becomes difficult to predict the waiting time for specific order at any prescribed service station. For better scheduling, orders must be scheduled with a provision of waiting at the various service stations they will pass through. Simulation helps us in accurately estimating such waiting times.

A good amount of work has been done for the development of inventory simulation models. There is fairly wide applicability and specialized use of such models as determination of optimal reorder level and lot size under probabilistic demand and lead time, optimal review period and ordering policy for continuous review inventory models.

A large number of network simulation models have also been developed so far. For example, we can determine the critical path even if we are given randomly selected activity-time for each activity. Repeating this process large number of times, the probability distribution for project completion time can be determined. Also, we can determine the probability that each given activity is on the critical path.

Many other applications of simulation models include the following :

(*i*) *Financial studies involving risky investments.*

(*ii*) *Military studies of logistics, support planning and weapon system effectiveness.*

(*iii*) *Testing of decision rules for hospital admission and operating policies.*

(*iv*) *Studies of individual and group behaviour.*

Simultation technique is the best computer-based tool to study the behaviour of the optimal solution due to small changes in the system's parameters.

Simulation is a numerical solution method that seeks optimal alternatievs (strategies) through a trial and error process. The simulation approach can be used to study almost any problem that involves uncertainty, *i.e.*, one or more decision variables can be represented by a probability distribution, like decision-making. However simulation approach requires an analogous physical model to represent mathematical and logical relationship among variables of the problem under study. After constructing the desired model, the simulation approach evaluate each alternative (measure of performance) by generating a series of values of random variable on paper over a period of time within the given set of conditions or criteria. This process of generating series of values one after another to understand the behaviour of the system (opertional informations) is called *executing (running or experimenting)* model on computer.

The precise definition of simulation given by **Shannon** is :

Simulation is process of designing a model of a real system and conducting experiments with this model for the purpose of understanding the behaviour (within the limits imposed by a criterion or set of criteria) for the operation of the system.

Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationship necessary to describe the behaviour and structure of a complex real-world system over extended periods of time. **–Naylor**

Simulation is the fast and relatively inexpensive method of performing 'experiments' on the computers. For example,

(i) In inventory control, the problem of determining the optimal replenishment policy arises due to the probabilisitic (stochastic) nature of demand and lead time. Thus, instead of trying manually the three replenishment alternatives for each level of demand and lead time for a period of one year and then selecting the best one, we process on the computer and obtain the results in a very short time at a very small cost.

(ii) In queuing theory the problem of balancing the cost of waiting against the cost of the idle time of service facilities in the system arises due to the probabilistic nature of the inter-arrival times of customers and the time taken to complete service to the customer. Thus, instead of trying out in actual manually with data to design a single server queuing system, we process the data on computers and obtain the expected value of various characteristics of the queueing system such as idle time of servers, average waiting time, quene length etc.

Unlike various analytical methods there is no rule to guide the formulation of simulation models. Each application of simultaion is different from the other and adhoc to a large extent.

Q. 1. What is meant by Monte-Carlo method of simulation ? Discuss its scope.

2. Explain Monte-Carlo method pointing but its uses in Operations Research.

3. Which computer based tools are appropriate to study the behaviour of the optimal solution due to small changes in the systems parameters ? What is the formal name given to this kind of exploration ? **[IGNOU 98 (June)]**

4. What are short path algorithms ? **[AIMS (MBA) 2002]**

17.15. SUMMARY OF SIMULATION PROCEDURE

The following steps are performed for simulation procedure :

1. Select the measure of effectiveness.
2. Decide the variables which influence the measure of effectiveness significantly.
3. Determine the cumulative probability distribution for each variable in *Step 2.*
4. Choose a set of random numbers.
5. Consider each random number as a decimal value of the cumulative probability distribution.
6. Insert the simulated values thus generated into the formula derived from the chosen measure of effectiveness.
7. Repeat *Steps 5* and *6,* until sample is large enough for the satisfaction of the decision maker.

Q. 1. What is Simulation Technique ? **[Kanpur M.Sc. (Math.) 96]**

2. Explain Simulation Modelling.

3. Distinguish between mathematical models and simulation models.

4. What are Simulation and Mont-Carlo methods used in Operations Research ? **[Meerut (Stat..) 90]**

5. Write a short note on simulation methods in sampling ? **[Meerut (I.P.M.) 91]**

6. Explain Monte-Carlo method of simulation.How would you use the method of simulation :
(*i*) exact probability distribution, (*ii*) empirical probability distribution ?
Indicate some queueing models where you would use simulation.

7. Suggest any two probability distributions applied to discrete random variables. Give expressions for their probability density functions. Discuss the situations where they can be qpplied.

8. Discuss the Monte-Carlo method of solving a problem illustrating it by outlining a procedure to solve a specified problem of your choice by the same.

9. State two major reasons for using simulation. Explain the basic steps of Monte-Carlo simulation. Briefly describe the application in finance and accounting.

10. Briefly explain the Monte-Carlo simulation with suitable example.

11. Define simulation. Why is simulation used ? Give one application area where this technique is used in practice.

12. "Simulation is the process of carrying out sampling experiments on the model of the system then the system itself". Discuss. **[Delhi MBA (FT) Dec. 94]**

13. Write short notes on simulation and its application

14. "Simulation is an especially valuable tool in a situation where the mathematics needed to describe a system realistically is too complex to yield analytical solutions" Elucidate. **[C.A. (Nov) 91]**

15. Explain with illustrations, how Monte-carls methods are useful in operations Research. **[Garhwal M.Sc. (Stat.) 95]**

SELF EXAMINATION PROBLEMS

1. An automobile production line turns out about 100 cars a day, but deviations occur owing to many causes. The production is more accurately described by a probability distribution as given below :

Production (per day) :	95	96	97	98	99	101	102	102	103	104	105
Probability :	0.03	0.05	0.07	0.10	0.15	0.20	0.15	0.10	0.07	0.05	0.03

Finished cars are transported across the bay at the end of each day by ferry. If the ferry has space only for 101 cars, what will be the average number of cars waiting to be shipped ?

2. An Airline has 15 flights leaving a base per day, each with a hostess. The airline keeps three hostesses in reserve so that they may be called in case the scheduled hostess for a flight reports sick. The probability distribution for daily number of sick hostesses is as follow :

Number of sick :	0	1	2	3	4	5
Probability	0.20	0.25	0.20	0.15	0.10	0.10

Use Monte-Carlo method to estimate the utilization of reserve hostesses and also the probability that at least one flight will be cancelled in a day because of non-availability of a hostess. Compare with exact results.

3. An owner of a petrol pump with single attendant wishes to perform a simulation of his operations to see whether any improvement is possible. He studied the system and found that an average of 6 customers arrive for service with random arrival times and form a queue, and the attendant provides service for exactly 9 minutes. For simulating the arrival times of customers, he has selected 10 random numbers with expected length of interval equal to one as :

3.62, 1.78, 1.84, 1.31, 1.27, 0.14, 1.71, 0.77, 0.97, 1.32

Find : (*i*)The total idle time for the attendant (*ii*) Total waiting time for the customers; and (*iii*) Maximum queue length during this period.

If the service time is reduced to 6 minutes, what is the quality of the service ?

4. A tourist car owner has 25 taxis in operation. He keeps three drivers as reserve to attend the calls, in case the scheduled driver reports sick. The probability distribution of sick drivers is as follows :

Number of sick :	0	1	2	3	4	5
Probability	0.20	0.25	0.20	0.15	0.12	0.08

Use Monte-Carlo method to estimate the utilization of reserve drivers and the probability that at least one taxi will be off the road due to non-availability of a driver. Compare with the correct answers.

5. A machine shop has 30 machines. a sample of 73 break-downs of machines is according to the following distribution :

Time between breakdowns (hrs.) :	10	11	12	13	14	15	16	17	18	19
Frequency :	4	10	14	16	12	6	4	3	3	1 (Total = 73)

A study of time required to repair the machines by one mechanic yields the following distribution :

Repair Time (hours) :	8	9	10	11	12	13	14	15	16	17	18
Frequency :	2	3	8	16	14	12	8	5	3	1	1 (Total = 73)

(*i*) Convert the distribution to cumulative probability distribution.

(*ii*) Using a simulated sample of 20, estimate the average per cent machine waiting time and the average per cent idle time of repair mechanic.

6. A gas transport company controls pipe-lines between several natural gas fields and out of state distributors. The company has a 1,00,000 unit storage capacity. Because of federal regulations, the company receives either 40,000 or 60,000 units per day. There is no equal probability of either quantity being shipped on a given day. The actual demand for natural gas is given by the following table of relative frequencies :

Daily Demand :	25,001 – 45,000	45,000 – 55,000	55,000 – 65,000
Probability :	0.3	0.3	0.4

(a) What is the expected daily demand ?

(b) Construct a model that can be used to simulate the company's daily receiving, storage and shipping activities.

7. Bharat Transport Company is considering discontinuing its leasing of 12 pickup and delivery trucks. If it does this, it will have to buy 12 trucks right away and then buy replacement truck in the future, one by one, as the old ones wear out. The question Bharat Transport Company's manager wants to answer is; how many trucks they may have to buy during the next five years (including the initial 12 trucks) in order to keep 12 trucks in operation all the time. At the moment, they are no concerned with the fact that at the end of five years they will have 12 trucks on hand. Some of which will be relatively

new and some of which will be old. The only question is : how many will have to be purchased so they can plan their cash requirements accordingly. The following table shows a history of truck life :

Truck life (months) :	12	15	18	21	24	27	30
% of Trucks which have worn out :	5	10	20	25	30	5	5

8. An airline company is opening a city reservation service. A passenger making reservation will be able to telephone the office and place his request. The company wishes to decide how many telephone lines to phone and personnel expenses that vary with the number of lines. But, it also wishes to compare the level of service for the several alternatives,i.e., the number of lines. Assuming the following operating characteristics of the system, simulate the experience of 20 calls when there are : (a) just one line, (b) just two Lines :

(*i*) The average time between calls. (*ii*) The average service time of those calls that are answered.

(*iii*) The number of calls not answered. (*iv*) The fraction of time when all the lines are busy.

(*v*) The average length of time during which all the lines are busy.

The operating characteristics :

Initial Position : All the lines are free at the start.

Inter-arrival Time : Toss four unbiased coins and assign an inter-arrival time as follows :

No. of Heads :	4	3	2	1	0
Inter-arrival Time (in minutes) :	3	2	1	4	5

Service Time (*Occuring for a call if not all the lines are busy*) : Toss two unbiased coins and assign a service time as follows : -

N.o. of Heads :	2	1	0
Service Time (in minutes) :	1	2	2

[You need not toss coins, and instead, use the table of Random Numbers].

9. XYZ Company operates an automatic car-wash facility in a city. The manager is concerned about the long lines of cars that build up waiting for service. The service time for the system is mechine pace and thus constant, and the manager has the opportunity to decrease the service time by increasing the speed of the conveyer that pulls the cars through the system. Of course, the quicker the pace, the lower the quality of the car wash. Tjhe manager wants to study the effects of setting the system for a 2-minutes car wash. The following data on customer arrivals have been gathered.

Interarrival time (Minutes) :	1	2	3	4	5
Number of occurrence :	136	34	102	51	17

(a) Compare the traffic density for a service rate of 2 minutes per car.

(b) Simulate the arrival of 20 customers. The doors open at 8.00 a.m. Compute the average waiting time per customer, the average time spent in the system, the percentage of time the car wash is idle, and the maximum queue length.

10. The materials manager of a firm wishes to determine the expected (mean) demand for a particular item in stock during the reorder lead time. This information is needed to determine how far in advance to reorder, before the stock level is reduced to zero. However, both the lead time, in days, and the demand per day for the item are random variables, described by the probability distribution given below :

Lead time (days)	Probability	Demand/day	Probability
1	0.5	1	0.1
2	0.3	2	0.3
3	0.2	3	0.4
		4	0.2

Manually simulate the problem for 30 reorders, to estimate the demand during lead time.

11. A retail store distributes catalogues and takes orders by telephone. Distributions for intervals between incoming calls and the length of time required to complete each call are given below. The store management has determined that they want a probability of no more than 5% that a caller will have to wait longer that 10 seconds for the telephone to be answered. Use simulation to determine how many sales representatives should be available to answer incoming calls.

Interval between incoming calls (seconds)	Probability	Length of call (seconds)	Probability
10	0.08	60	0.07
12	0.11	65	0.12
14	0.14	70	0.18
16	0.16	75	0.16
18	0.14	80	0.15
20	0.12	85	0.12
22	0.08	90	0.08
24	0.07	95	0.06
26	0.04	100	0.06
28	0.04		
30	0.02		

12. The manager of a warehouse is interested in designing an inventory control system for all of the products in stock. The demand for the product comes from numerous retail outlets and orders arrive on a daily basis. The warehouse receives its stock from the factory, but the lead time is not constant. The manager wants to determine the best time to relase orders to the factory so that stock-outs are minimized yet inventory holding costs are at acceptable level. Any orders from retailers not supplied on a given day constitute lost demand. Based on a sampling study the following data are available :

Daily Demand	Probability	Lead time (days)	Probability
1	0	1	0.20
2	0.20	2	0.50
3	0.50	3	0.30
4	0.20		
5	0.10		

Two alternative ordering policies have been proposed :

Policy 1 : Whenever the inventory level drops below 6 units (reorder point), order 10 units (lot size).

Policy 2 : Whenever the inventory level drops below 10 units, order 10 units.

(a) Simulate each of these two policies for 20 days. Assume that you have 12 units in inventory at the start of the simulation.

(b) Compare and contrast the outcomes for each of these two policies.

13. A firm has a single channel service station with following arrival and service time probability distributions :

Arrivals (Min.)	Probability	Service time (Min.)	Probability
1.0	0.35	1.0	0.20
2.0	0.25	1.5	0.35
3.0	0.20	2.0	0.25
4.0	0.12	2.5	0.15
5.0	0.08	3.0	0.05

The customer's arrival at the service station is a random phenomenon and the time between the arrivals varies from one minute to five minutes. The service time varies from one minute to three minutes. The queueing process begins at 10.00a.m. and proceeds for nearly 2 hours. An arrival goes to the service facility immediately, if it is free. Otherwise it will wait in a queue. The queue discipline is first come first served. If the attendant's wages are Rs. 8 per hour and the customer's waiting time costs Rs. 9 per hour, then would it be an economical proposition to engage second attendant ? Answer on the basis of Monte-Carlo simulation techinque.

14. An automatic machinery company receives a different number of orders each day and the orders vary in the time required to process them. The firm is interested in determining how many machines it should have in the department to minimize the combined cost of machine idle and order waiting time. The firm knows from the past experience the average number of orders per day and the average number of hours per order which are as follows :

Number of orders per day :	0	1	2	3	4	5		
Probability :	0.10	0.15	0.25	0.30	0.15	0.05	5	0.05
Hours/order :	5	10	5	20	25	30	35	40
Probability :	0.05	0.05	0.10	0.10	0.20	0.25	0.15	0.10

Cost/hour of idle machine time = Rs. 4.00, cost/hour for orders waiting = =Rs. 6.00

Assuming 24 hours working in three shifts, solve the problem using simulation.

15. The State Distribution Corporation has its customers sent their own purchase orders. In the past, the arrival of these purchase orders per day has approximated a normal distribution with a mean of 50 and a standard deviation of 6. In tems of the probability of occurrence, the following is being indicated.

Number of purchase orders	26-32	32-38	38-44	44-50	50-56	56-62	62-68	68-74
Probability of occurrence	0.5	2.0	13.0	36.0	33.0	13.0	2.0	0.5

Develop a Monte-Carlo simulation of the number of purchase orders per day to be expected for a particular month. If the firm can process only 41 orders per day, how many days in the month will the firm be behind schedule?

16. The management of a company is considering the problem of marketing a new product. The investment or the fixed cost required in the project is Rs. 25,000. There are three factors that are—uncertain selling price, variable cost and the annual sales volume. The product has a life of only one year. The management has the past data regarding the possible levels of the three factors.

Unit Selling Prices(Rs.)	Probability	Unit Variable Cost (Rs.)	Probability	Sale Volume (Units)	Probability
40	0.30	20	0.10	3,000	0.20
50	0.50	30	0.60	4,000	0.40
60	0.20	40	0.30	5,000	0.40

Using Monte-Carlo simulation technique, determine the average profit from the said investment on the basis of 20 trials.

17. A trader has studied his varying monthly sales and monthly expenses (including the value of goods) and has arrived at the following empirical distributions :

Monthly sales Rs. ('000)	Probability	Monthly expenses Rs. ('000)	Probability
15	0.30	12	0.15
16	0.25	13	0.20
17	0.15	14	0.25
18	0.15	15	0.20
19	0.10	16	0.15
20	0.05	18	0.05

(a) The trader at the beginning of the year has Rs. 2,000 in the bank. Simulate his sales and expenses over a year (two times). Assume that the trader can avail temporary overdraft facilities to cover any negative balance.

(b) How much money does the trader have at the end of the year ?

18. A certain maintenance facility is responsible for the upkeep of five machines. The machines, which fail frequently, must be repaired as soon as possible to maintain as high a productive capacity of the production system as possible. Management is concerned about the average down time per machine and is considering an increase in the capacity of the maintenance facility. The following distributions have been developed from historical data.

Time between break down (days)	Probability	Repair time (days)	Probability
2	0.05	1	0.40
3	0.10	2	0.50
4	0.15	3	0.10
5	0.40		
6	0.20		
7	0.10		

Simulate the failure and repair of 10 machines. Begin by determining the time of the first break for each of the 5 machines. Sequence the machines through the repair facility on a first come first service basis. If there is more than one machine waiting to be repaired, arbitrarily choose one to repair next. After a machine has been repaired, determine its next time of breakdown and continue until you have repaired 10.

19. Assume a firm desires to evaluate cash flows for planning purposes. In this firm, orders are received at the beginning of the month for delivery at the end of the month. Monthly sales and the associated probabilities are as follows :

Monthly sales :	100 units	125 units	150 units
Probability :	60%	30%	10%

The unit selling price is Rs. 250. Cash is received either in the month of sale or the following month. From previous experience the firm has found that if collection is delayed until the following month, 5% of the receivables are not collectible.

Collection and associated probabilities are :

Collection made :	Current month	Following month
Probability :	70%	30%

Fixed cash disbursements are Rs. 15,000 a month. Variable cost disbursements are Rs. 100 per unit produced (sold). If overtime (OT) is required, the estimated cash disbursements are :

Production	Probabilities for	
	OT not required	OT required
100 units	0.8	0.2
125 units	0.7	0.3
150 units	0.5	0.5

OT costs are 50 per cent more on variables costs.

Attempt a simulation for 12 months to establish a cash flow pattern using random numbers from the following table :

9548	5099	9747	3755	4162
4552	6291	1830	7263	7010
9969	8265	1572	7705	1352
3512	4191	4570	4826	3140

State assumptions.

20. Using random numbers to simulate a sample, find the probability that a packet of 6 products does not contain any defective product, when the production line produces 10% defective products. Compare your answer with the expected probability. **[I.C.W.A. (dec.) 90]**

21. The materials manager of a company is interested in determining the reorder point for an item, the pattern of demand for which data is as given below :

No. of units per day	:	0	1	2	3	4	5
No. of days on which the demand occurred	:	5	9	16	38	23	9

Past experience indicates that there are fluctuations in the lead time for procurement of item. The following data are available for the records for the last 30 orders :

Lead time (weeks)	:	1	2	3
No. of times the specified lead time occurred	:	18	7	5

The management policy is to ensure that the proportion of stockouts should not exceed 5%. Illustrate how the simulation approach can be used to determine the re-order point. (Analysis for 20 orders is adequate)

22. A company trading in motor vehicle spares wish to determine the level of stock it should carry for the items in its range. Demand is not certain and there is a lead time for stock replenishment for one item x, the following information is obtained.

Demand (units/day)	:	3	4	5	6	7
Probability	:	0.1	0.2	0.3	0.3	0.1

Carrying cost (per unit per day) = 20 paise
Ordering cost (per order) = Rs. 5
Lead time for replenishment = 3 days

Stock in hand at the begining of the simulation exercise was 20 units.

You are required to carry out a simulation run over a period of ten days with the objective of evaluating the following inventory rule :

Order 15 units when present inventory plus any outstanding order falls below 15 units.

The sequence of random numbers used is 0, 9, 1, 1, 5, 1 ,8, 6, 3, 5, 7, 1, 2, 9, using the first number for day one.

Your calculation should include the total cost of operating this inventory rule for 10 days. **[I.C.W.A (June) 91]**

23. A ware-house manager, has been disturbed by what appears to be a larger than expected backlog of orders to be filled. He employs a clerk to fill orders one orders are received according to the following empirically established distribution.

Interarrival Time (in Minutes) :	30	45	60	75	90
Frequency :	15	20	35	20	10

The distribution of service time necessary for a single clerk to fill orders is as follows :

Service Time (in Minutes) :	60	90	120	150	180
Frequency :	10	30	30	20	10

The clerk works alone taking orders from a single inbox. The managers wants you to simulate the hequired time for filling 20 orders. **[Delhi MBA (FT) Dec.94]**

24. A company manufactures around 200 mopeds. Depending upon the availability of raw materials and other conditions, the daily production has been varying from 196 mopeds to 204 mopeds, whose probability distribution is as given below :

Production per day :	196	197	198	199	200	201	202	203	204
Probability :	0·05	0·09	0·12	0·14	0·20	0·15	0·11	0·08	0·06

The finished mopeds are transported in a specially designed three storyed lorry than can accommodate only 200 mopeds. Using the following 15 random numbers 82, 89, 78, 24, 53, 61, 18, 45, 04, 23, 50, 77, 27, 54, 10, simulate the process to find out :

(i) What will be the average number of mopeds, waiting in the factory ?

(ii) What will be the average number of empty spaces on the lorry ? **[C.A. May 99]**

[Hint. The random numbers are established as in table below :

Production per day	Probability	Cumulative probability	Random number
196	0·05	0·05	00–04
197	0·09	0·14	05–13
198	0·12	0·26	14–25
199	0·14	0·40	26–39
200	0·20	0·60	40–59
201	0·15	0·75	60–74
202	0·11	0·86	75–85
203	0·08	0·94	86–93
204	0·06	1·00	94–99

Based on the 15 random numbers given we simulate the production per day as shown in table 2 below :

S. No.	Random no.	Production per day	No. of mopeds waiting	No. of empty spaces in the lorry
1	82	202	2	
2	89	203	3	
3	78	202	2	
4	24	198		2
5	53	200	0	0
6	61	201	1	
7	18	198		2
8	45	200	0	0
9	04	196		4
10	23	198		2
11	50	200	0	0
12	77	202	2	
13	27	199		1
14	54	200	0	0
15	10	197		3
Total			10	14

Results : Average no. of moped waiting in the factory = 10/15 = 0·67 per day

Average no. of empty spaces in the lorry = 14/15 = 0·93 days.

25. In a two-station assembly line of a product, the operator at station *A* completes approximately half of the assembly and then sets it on a section of a conveyer where it rolls down to station *B* where the assembly is completed. The distribution of assembly times at stations *A* and *B* are as follows :

Assembly time at station *A*	Frequency	Assembly time at station *B*	Frequency
0·2	3	0·2	2
0·3	2	0·3	12
0·4	25	0·4	18
0·5	30	0·5	25
0·6	17	0·6	19
0·7	10	0·7	13
0·8	8	0·8	11
0·9	5		

By using Monte-carle simulation model, determine the average inventory of assemblies between stations *A* and *B*. Perform ten simulations. Take following random numbers :

For station *A* :	78	46	33	26	96	58	98	27	27	72
For station *B* :	71	92	34	93	13	59	75	39	57	10

[Delhi (M.B.A.) Dec. 96]

26. The wholesaler dealing in stationary items wants to determine the order size for desk calenders. The demand and lead times are probabilisitc and their distributions are given below :

Demand/week (thousand)	Probability	Lead time (weeks)	Probability
0	0·2	2	0·3
1	0·4	3	0·4
2	0·3	4	0·3
3	0·1		

The cost of placing an order is Rs. 50 per order and the holding cost for 1,000 calendars is Rs. 2 per week. The shortage cost is Rs. 10 per thousand. The manager is considering the policy :

Reorder point = 2,000 units

Order quantity = Difference between the current inventory balance and maximum replenishment level of 4,000.

Using the random numbers given below *simulate* the policy for 20 weeks' period assuming that :

(a) the beginning inventory is 3,000 units, (b) no back orders are permitted, (c) each order is placed at the beginning of the week following the drop in inventory level to (or below) the reorder point, (d) the replenishment orders are received at the beginning of the week.

Random numbers :

31, 29, 70, 53, 86, 83, 32, 78, 26, 64, 45, 12, 99, 58, 52, 43, 84, 38, 40, 41, 19, 87, 83, 73, 13

(These random numbers should be used for generating both demand and lead time probabilities.)

[IGNOU (M.B.A.) Dec. 98]

27. A tourist agency has presently a fleet of three high quality cars, which it rents out to customers on a day to day basis. The number of customers is probabilistic variable as per pattern given below. Note that each customer will demand only one car.

No. of cars demanded/day :	0	1	2	3	4
Probability :	0·10	0·30	0·35	0·15	0·10

Further, the number of days a customer may want a car may also vary. The probability pattern is :

No. of days :	1	2	3
Probability :	0·50	0·35	0·15

The number of days of demand are independent of the customers. Cars are hired for the day(s) in the morning and returned late in the evening at the end of the day (or 2 days etc.). The cars returned in the evening are available for re-rental for the next day.

Whenever is rented out then the agency earns a profit of Rs. 400 per day of rental. Whenever a car is idle there is a loss of Rs. 90 per day if idleness. Whenever a demand cannot be met then the agency suffers a goodwill loss estimated at Rs. 700 per demand.

The agency experiences that with a pool of three cars substantial goodwill loss is incurred. It wants to decide whether it would be **worthwhile** to acquire an additional car. Presently with three cars the average daily net profit works out to Rs. 380. A new car will be added only if the average daily earnings increases to at least 500.

Using the random numbers given below, simulate the operations for a period of 12 days and assess the advisability of acquisition of a fourth car. Assume that on day one all cars are available for renting.

Random Digits :

(use the digits continuously and in the equal order in which the probabilistic will occur in real time. Start at the first row and proceed row-wise)

7826	4397	9431	6287	3389	0807	6529
7203	7972	9853	8795	1173	6152	4475
1264	4806	0471	6635	2427	4320	0042

[Narsee Monjee Institute of Management (PGBDBA) 94, 98]

28. The material's manager of a company is interested in determining the re-order point for an item, the pattern of demand for which it is given below :

No. of Units (per day)	0	1	2	3	4	5
No. of Days on which the demand occurred	5	9	10	38	23	9

Past experience indicates that there are fluctuations in the lead time for procurement of the item. The following data are available from records for the last 30 orders

Lead Time (weeks)	1	2	3
No. of times the specified lead Time Occurred	181	7	5

The management policy is to ensure that the proportion of stockouts should not exceed 5 per cent. Illustrate how the simulation opproach can be used to detemine the re-order point (Analysis for 20 orders is adequate).

[Delhi Univ. (MBA) 1999]

OBJECTIVE QUESTIONS

1. An advantage of simulation as opposed to optimization is that
(a) several options of measure of performance can be examined.
(b) complex real-life problems can be studied.
(c) it is applicable in cases where there is an element of randomness in a system.
(d) all of the above.

2. The purpose of using simulation technique is to
   (a) simulate a real world situation.
   (b) understand properties and operating characteristics of complex real-life problems.
   (c) reduce the cost of experiment on a model of real situation.
   (d) all of the above.
3. Which of the following is not the special purpose simulation language ?
   (a) BASIC. (b) GPSS. (c) GASP. (d) SIMSCRIPT.
4. As simulation is not and alytical model, therefore result of simulation must be viewed as
   (a) unrealistic. (b) exact. (c) approximation. (d) simplified.
5. While assigning random numbers in Monte Carlo simulation, it is
   (a) not necessary to assign the exact range of random number interval as the probability.
   (b) necessary to develop a cumulative probability distribution.
   (c) necessary to assign the particular appropriate randem numbers.
   (d) all of the above.
6. Analytical results are taken into consideration before a simulation study so as to
   (a) identify suitable values of the system parameters.
   (b) determine the optimal decision.
   (c) identify suitable values of decision variables for the specific choices of system parameters.
   (d) all of the above.
7. Biased random sampling is made from among alternatives which have
   (a) equal probability. (b) unequal probability.
   (c) probability which do not sum to 1. (d) none of the above.
8. Large complicated simulation models are appreciated because
   (a) their average costs are not well-defined.
   (b) it is difficult to create the appropriate events.
   (c) they may be expensive to write and use as an experimental device.
   (d) all of the above.
9. Simulation should not be applied in all cases because it
   (a) requires considerable talent for model building and extensive computer programming efforts.
   (b) consumes much computer time.
   (c) provides at best approximate solution to problem.
   (d) all of the above.
10. Simulation is defined as
   (a) a technique which uses computers.
   (b) an approach for reproducing the processes by which events by chance and changes are created in a computer.
   (c) a procedure for testing and experiments on models to answer what if. . .,then so and so ... types of questions.
   (d) all of the above.

Answers

1. (d) **2.** (d) **3.** (a) **4.** (c) **5.** (b) **6.** (c) **7.** (b) **8.** (c) **9.** (d) **10.** (d).

* * *

18

Decision Theory

18.1. INTRODUCTION

Decision making is an integral part of management planning, organizing, controlling and motivation processes. The decision-maker selects one strategy (course of action) over others depending on some criteria, like utility, sales, cost or rate of return. The specific combination of goals is not entirely depending on the decision-maker. That is, the value system is usually modified by other interested groups, like stock holders, employers, unions, creditors, government, etc.

In fact, the decision should be made whenever the organization or an individual faces a problem of decision making or dis-satisfied with the existing decisions or when alternative selection is specified. Then it becomes necessary to develop rational decision making to improve decision making abilities. This can best be done by analysing previous decisions and better understanding of nature and decision process.

Decision theory provides a method for rational decision-making when the consequences are not fully deterministic. Dicision-maker has to apply various methods to decision problems. The decision theory identifies the best alternative or course of action for specific activity. Decision theory provides a framework for better understanding of the decision situation and for evaluating alternatives when the criteria are not defined. Fig. 18.1 indicates the relationship between decision theory and decision making.

If a decision is to be made on the basis of one of the decision criteria, then the selection of an alternative course of action is a direct effect of decision theory. But, if alternative course of actions is compared with the cosiderations other than the so called pure criteria, then decision theory helps us in the evaluation of alternatives.

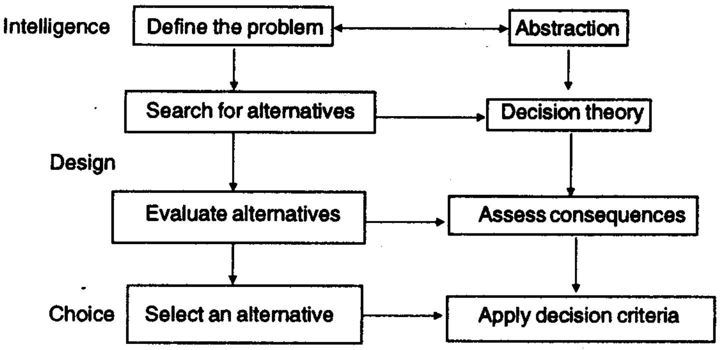


Fig. 18.1.

18.2. TYPES OF DECISIONS

In general, decisions can be classified into three categories :

1. Strategic Decisions. These decisions are concerned with external environment of the organization. For example, decision of selection of product-mix which a firm will produce and the markets to which it will sell are under this category.

2. Administrative Decision. This is concerned with structuring and acquisition of the organization's resources so as to optimize the performance of the organization. Selection of distribution channels, location of facilities, etc. are some of the examples of such decisions.

3. Operating Decision. This is primarily concerned with day-today operations of the organization such as pricing, production scheduling, inventory levels, etc.

18.3. COMPONENTS OF DECISION MAKING

To formulate the problem of decison making correctly, we must know what a problem is, and to understand the nature of the problem. Various components of the problem which a person should know are briefly discussed as here.

1. The decision maker. Most obvious component is the fact that someone belonging to some group must have the problem. Thus individual or group is dissatisfied with some aspect of the state of affairs and consequently wants to make a decision with regard to altering it, and referred to as the decision maker. He may, also be referred to as policy-maker or executive. If the decision maker controls the operations of an organized system of men and/or machines.

2. Objectives. In order to have a problem, the decision-maker, he must know something other than what he has, *i.e.* he must have some objectives which he has not obtained to the degree he desires.

3. The system, or environment. The decision-maker has the problem in an environment or setting that contains or lacks various resources. In the type of problem in which O.R. involved, this environment is an organized system usually embracing machines as well as man.

4. Alternative courses of action. A problem cannot exist unless the decision maker has a choice from among at least two alternative courses of action or policies. A problem always involves a question : what to do ? This question becomes a problem only when alternative courses of action are available.

5. Choices must have unequal efficiencies for the desired outcomes. The courses of action available to him must provide some chance of obtaining his objectives, but they cannot provide the same chance. Otherwise, his choice would not matter.

Thus, a problem exists if above five conditions are satisfied; and an individual is said to have a problem if he does not know what course of action is the best one and he wants to know it.

18.4. DECISION MODELS

One of the primary function of management is to make decisions that determine the future course of action for the organization involving short-term or long-term consequences. Decision models are classified into various categories, depending upon their nature and complexity, such as allocation models which are used to allocate resources into various activities so as to optimize the given objective function subject to certain restrictions. However, there are some elements as listed below which are common to all kinds of decisions.

1. Goals to be achieved. The objectives which the decision-maker wants to achieve by his actions.

2. The decision-maker. The decision-maker refers to an individual or a group of individuals responsible for making a choice of an appropriate course of action amongst the available courses of action.

3. Courses of action. Courses of action sometimes called *actions,* or *decision alternatives.* For a specified problem, all possible courses of action should be included. The number of possible courses of action may be large or small, but these are under the control of decision-maker, *i.e.* decision-maker determines which courses of action are possible.

A course of action can have a numerical description such as stocking of 150 units of a particular item or non-numerical description, *i.e.*, a decision to build up a stockpile in anticipation of a possible strike.

4. States of nature. Before applying decision theory, we must develop an exhaustive list of possible future events. However, decision-maker has no direct control over the occurrence of particular event. Such future events are referred to as *states of nature* and it is assumed that these are mutually exclusive and collectively exhasutive.

The state of nature can have a numerical description such as demand of some units of a given item or a non-numerical description like employee's strike.

5. The preference or value system. This refers to the criteria that the decision-maker uses in making a choice of the best course of action. It could include maximization of income, utility, profit, etc.

*6. **Payoff.*** It is the effectiveness associated with specified combination of a course of action and state of nature. These are also known as *profits* or *conditional values.* For example, the conditional profit can also be of Rs. 15 associated with the action of stocking 20 units of an item when the outcome is a demand of 17 units of that item. Costs can be considered as negative profits.

*7. **Opportunity loss table.*** Opportunity loss is incurred due to failure of not adopting most favourable course of action or strategy. The opportunity loss values are determined separately for each state of nature or outcome by first finding the most favourable course of action for that state of nature or outcome and then taking the difference between payoff value for given course of action and the payoff value for the best possible course of action which could be chosen.

*8. **Payoff table.*** For a problem, a payoff table, lists the states of nature which are mutually exclusive and collectively exhaustive and a set of given courses of action (strategies). For each combination of state of nature and course of action, the payoff is determined. The weighted profit associated with the given combination of state of nature and course of action is obtained by multiplying the payoff for that state of nature and course of action by the probability of occurrence of the specified state of nature (outcome). The table shown below is one such type of payoff table. In this table, m states of nature are denoted by $O_1 , O_2 , \ldots , O_m$ with respect to n courses of action $S_1 , S_2 , \ldots , S_n$. For a specified combination of state of nature and course of action, the corresponding payoff is represented by a_{ij}.

Payoff Table

States of Nature	Courses of Action	
	S_1	$S_2 \ldots S_j \ldots S_n$
O_1	a_{11}	a_{12} .... a_{1j} .... a_{1n}
O_2	a_{21}	a_{22} .... a_{2j} .... a_{2n}
:	:	: : :
O_i	a_{i1}	a_{i2} .... a_{ij} .... a_{in}
:	:	: : :
O_m	a_{m1}	a_{m1} .... a_{m2} .... a_{mn}

18.5. TYPES OF ENVIRONMENT

Decision theory helps the decision-maker in selecting the best course of action from the available courses of action. The decision models are classified such that the type of information which is given about the occurrence of the various states of nature as well as depending upon the decision environment. Basically, there are four different states of decision environments : *certainty, uncertainty, risk,* and *conflict.*

1. Decision-making under certainty. This is an easiest form of decision-making. The outcome resulting from the selection of a particular course of action given with certainty. There is just one state of nature for each course of action and has probability 1. We are given complete and accurate knowledge of the consequence of each choice. Since the decision maker has perfect knowledge of the future and the outcome, he simply selects that course of action for which the payoff is optimum.

For example, the analysis of *cost, profit* and *volume* a decision problem under certainty, where the information regarding costs and profits is given with respect to volume of sales. Similarly, in linear programming problem, the amount of resources required and the corresponding unit profit/cost is given with certainty. Other techniques used for solving problems under certainty are : (i) *Input-output analysis,* (ii) *Breakeven analysis,* (iii) *Goal programming* (iv) *Transportation and Assignment methods,* (v) *Inventory models under certainty,* etc.

2. Decision-making under risk. Under the condition of risk, the various states of nature can be enumerated and the long-run relative frequency of their occurrence is assumed to be given. The information about the states of nature is probabilistic, *i.e.* decision-maker cannot predict which outcome (payoff) will occur as a result of his selecting a particular course of action. Since each course of action results in more than one outcome, it is not simple to find the exact monetary payoffs or outcomes for the several combination of courses of action and states of nature. But past experience or past records often enable the decision-maker to assign probability values to the likely possible occurrence of each state of nature. Knowing the probability distribution of states of nature, the best decision is to choose such course of action which has the largest expected payoff.

In this case, most widely used criterion for evaluating the alternative courses of action, is the *Expected Monetary Value (EMV)* or expected utility. The objective of decision-making under this condition is to optimize the expected payoff.

Q. Differentiate decision under certainty and decision under risk. **[Bhubneshwar (IT) 2004]**

18.5-1. The Expected Monetary Value (EMV)

The expected monetary value (*EMV*) for the specified course of action is the weighted average payoff, *i.e.* the sum of product of the payoff for the several combination of courses of action and states of nature multipled by the probability of occurrence of each outcome.

In the case of *i.e.* probabilistic situation each course of action can lead to a number of different possible outcomes. So the decision-maker is always interested to know two things : (i) *the average payoff or expected value of an outcome (or random variable)*, (ii) *risk involved in the particular course of action.* Both of these given informations can be obtained by using the concept of *expected value* (if measured in terms of monetary value, then *expected monetary value*) or *mathematical expectations.*

Let us now assume that it is possible to attach a measure of probability to each value assumed by the specified state of nature. Then the expected monetary value corresponding to each course of action is given by

$$EMV(S_i) = \sum_{j=1}^{n} P(O_j)\, a_{ij}$$

where, $S_i =$ course, of action, i

$P(O_j) =$ probability of occurrence of state of nature O_j

$n =$ number of possible states of nature,

$a_{ij} =$ payoff associated with course of action i, and state of nature O_j

Remark. The expected value of a continuous random variable is defined as :

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

18.5-2. Steps for Calculating EMV

In solving such problems, it is useful to develop a general method to problem formulation and solution. The various steps for calculating *EMV* are as follows :

Step 1. Define systematically the state of nature (mutually exclusive and collectively exhaustive) and the course of action.

Step 2. List the payoff associated with each possible combination of *course of action* and *state of nature* along with the corresponding probability of each state of nature.

Step 3. Calculate *EMV* for each course of action by multiplying the conditional profits or costs or losses (payoff) by the associated probabilities and then sum these weighted values for each course of action.

Step 4. On the basis of this decision, obtain the course of action corresponding to the optimal *EMV*.

18.5-3. The Expected Value of Perfect Information (EVPI)

In probabilistic situation, there is no control over the occurrence of given state of nature. However, what will happen if decision maker had exact information about the occurrence of particular state of nature. For example, in a case of single large competitor, the decision for raising the price of certain product, will depend upon the degree of reliability that the competitor will also match the price increase or will leave the price unchanged. In this case, the states of nature are the set of competitor's likely responses to a price increase and these state of nature (for which an estimation of subjective probabilities is needed) are not under the control of the manufacturer of the product. The course of action or strategy under consideration is an increase in the product price or an unchanged product price. Forecasting the possibility of price increase, suppose the competitor knows that he will not raise his price even if there is price rise. Then, the manufacturer, will have *completely reliable* or *perfect information* about the state of nature that would occur if he **noted** to raise the price of his product. With such perfect information, manufacturer would be able to choose the optimal course of action with certainty.

Definition. The *expected profit with perfect information* (EPPI) is the maximum attainable expected monetary value (EMV) based on perfect information about the state of nature that will occur. The expected profit with perfect information may be defined as the sum of the product of best state of nature corresponding to each optimal course of action and its probability.

Definition. *The expected value of perfect information* (*EVPI*) may now be defined as the maximum amount one would be willing to pay to obtain perfect information about the state of nature that would occur. *EMV** represents the *maximum attainable expected monetary value* given only the prior outcome probabilities, with no information as to which state of nature will actually occur. Therefore, perfect information would increase profit from *EMV** upto the value of *EPPI*. This increased amount is termed as *expected value of perfect information* (*EVPI*), *i.e.* $EVPI = EPPI - EMV^*$

Q. Brielly explain "expected value of perfect information" with example. **[C.A. (May) 92]**

18.5-4. The Expected Opportunity Loss (EOL)

Another useful way of maximizing monetary value is to minimize the expected opportunity loss or expected value of regret. The conditional opportunity loss (*COL*) or regret function for a particular course of action is determined by taking the difference between the payoff value of the most favourable course of action (i.e., having highest payoff) and some other courses of action which may be considered as loss due to losing the opportunity of choosing the most favourable course of action. Thus, opportunity loss can be obtained separately for each course of action by first obtaining the best state of nature for the prescribed course of action and then taking the difference between that best outcome (or conditional profit) and each outcome for those courses of action. The opportunity loss for each course of action is known as the conditional opportunity loss (*COL*).

After claculating the opportunity loss value for each course of action, the expected opportunity loss (*EOL*) for *i*th course of action S_i is then computed by :

$$EOL\,(S_i) = \sum_{j=1}^{n} COL\,(S_i\,,\,O_j)\;P(O_j)$$

where $COL\,(S_i\,,\,O_j)$ = conditional opportunity loss associated with course of action S_i and state of nature O_j

$P(O_j)$ = probability of occurrence of state of nature O_j

In other words, *EOL* denotes the expected difference between the payoff of right decision and the payoff of actual decision.

Remarks 1. The optimal course of action will always be the course of action that has the minimum *EOL* value, whether the objective is one of maximizing expected profit or minimizing expected cost.

2. The *EOL* value for any course of action is equal to *EPPI* minus the *EMV* for that course of action. That is, it is the loss or penalty due to lack of perfect information.

Illustrative Examples

Example 1. *An investor is given the following investment alternatives and percentage rates of return.*

	State of nature (Market conditions)		
	Low	**Medium**	**High**
Regular shares	7%	10%	15%
Risky shares	– 10%	12%	25%
Property	– 12%	18%	30%

Over the past 300 days, 150 days have been medium market conditions and 60 days have had high market increases.

On the basis of these data, state the optimum investment strategy for the investment.

[Nagpur (M.B.A.,) Feb. 99]

Solution. According to the given information, the probabilities of *low, medium* and *high* market conditions would be 0·30, 0·50 and 0·20 respectively. The expected pay-offs for each of the alternatives are calculated and shown in the table below :

Table 18.1 : Computation Of Expected Return

Market conditions	Prob.	Strategy		
		Regular Shares	Risky Shares	Property
Low	0·30	– 0·07 × 0·30	0·10 × 0·30	0·15 × 0·30
Medium	0·50	– 0·10 × 0·50	0·12 × 0·50	0·25 × 0·50
High	0·20	– 0·12 × 0·20	0·18 × 0·20	0·30 × 0·20
Expected Return		0·136	0·126	0·230

Since the expected return of 23% is the highest for property, the investor should invest in this alternative.

Example 2. *The following pay-off table is given :*

Acts	Event			
	E_1	E_2	E_3	E_4
A_1	40	200	−200	100
A_2	200	0	200	0
A_3	0	100	0	150
A_4	−50	400	100	0

Suppose that the probability of the events of this table are $P(E_1) = 0{\cdot}20$, $P(E_2) = 0{\cdot}15$, $P(E_3) = 0{\cdot}40$, $P(E_4) = 0{\cdot}25$, *calculate the expected pay-off and the expected loss of each action.* **(C.A., May 97)**

Solution.

Table 18.2 : Computation Of Expected Pay-off

		Conditional Pay-off Act				Expected Pay-off Act			
Event	Prob. (1)	A_1 (2)	A_2 (3)	A_3 (4)	A_4 (5)	A_1 (1) × (2)	A_2 (1) × (3)	A_3 (1) × (4)	A_4 (1) × (5)
E_1	0·20	40	200	0	−50	8	40	0	−10
E_2	0·15	200	0	100	400	30	0	15	60
E_3	0·40	−200	200	0	100	−80	80	0	40
E_4	0·25	100	0	150	0	25	0	37·5	0
Expected pay-off :						−17	**120***	52·5	90

The oppotunity loss or regret table can be obtained from the given pay-off table.

Example 3. *An **electrical manufacturing company** has seen its business expanded to the point where it needs to increase production beyond its existing capacity. It has narrowed the alternatives to two approaches to increase the maximum production capacity : (a) expansion, at a cost of Rs. 8 million, or (b) modernization at a cost of Rs. 5 million. Both approaches would require the same amount of time for implementation. Management believes that over the required payback period, demand will either be high or moderate. Since high demand is considered to be somewhat less likely than moderate demand, the probability of high demand has been setup at* 0·35. *If the demand is high, expansion would gross an estimated additional Rs. 12 million but modernization only an additional Rs. 6 million, due to lower maximum production capability. On the other hand, if the demand is moderate, the comparable figures would be Rs. 7 million for expansion and Rs. 5 million for modernization.*

(a) Calculate conditional profit in relation to various action- and outcome combinations and states of nature.

(b) If company wishes to maximize its expected monetary value, then should it modernize or expand ?

(c) Calculate the EVPI. *(d) Construct the conditional opportunity loss table and also calculate EOL.*

Solution. (a) Defining the state of nature or outcome (over which the company has no control) and course of action (company's possible decision), let

States of nature : O_1 = *high demand,* O_2 = *moderate demand*

Courses of action : S_1 = *expand,* S_2 = *modernize.*

Since the probability that the demand is high (outcome O_1) is estimated to 0·35 , the probability of moderate demand (outcome O_2) must be $(1 - 0{\cdot}35) = 0{\cdot}65$. The calculations for conditional profit values are as follows :

Table 18.3 Conditional Profit (Million Rs.)

State of Nature O_j	Courses of Action	
	S_1 (Expand)	S_2 (Modernize)
O_1 (high demand)	$12 - 8 = 4$	$6 - 5 = 1$
O_2 (moderate demand)	$7 - 8 = -1$	$5 - 5 = 0$

(b) The payoff table 18·1 can be rewritten as follows along with the given probabilities of states of nature.

Table 18·4. Conditional Profit (Million Rs.)

State of Nature O_j	Probability $P(O_j)$	Curses of Action	
		S_1 (Expand)	S_2 (Modernize)
O_1	0·35	4	1
O_2	0·65	– 1	0

The calculation of *EMV*'s for courses of action S_1 and S_2 are given below :

$$EMV(S_1) = (0{\cdot}35)(4) + (0{\cdot}65)(-1) = 1{\cdot}40 - 0{\cdot}65 = \text{Rs. } 0{\cdot}75 \text{ million}$$

$$EMV(S_2) = (0{\cdot}35)(1) + (0{\cdot}65)(0) = 0{\cdot}35 = \text{Rs. } 0{\cdot}35 \text{ million}$$

To maximize *EMV*, the company must expand course of action S_1. The *EMV* of the optimal course of action is generally denoted by *EMV* *, therefore,

$$EMV^* = EMV(S_1) = \text{Rs. } 0{\cdot}75 \text{ million}$$

(c) To compute *VEPI*, we shall first calculate *EPPI*. For calculating *EPPI*, we choose the optimal course of action for each state of nature, multiply its conditional profit by the given probability to get weighted profit and then sum these weights as shown in the following table.

Table 18·5 Profit of Optimal Course of Action

State of Nature O_j	Probability $P(O_j)$	Optimal Course of Action	Conditional Profit	Weighted Profit
O_1	0·35	S_1	4	$4 \times 0{\cdot}35 = 1{\cdot}40$
O_2	0·65	S_2	0	$0 \times 0{\cdot}65 = 0$
			EPPI	1·40

The optimal *EMV* * is Rs. 0·75 million corresponding to the course of action S_1. Then

$$EVPI = EPPI - EMV(S_1) = 1{\cdot}40 - 0{\cdot}75 = \text{Rs. } 0{\cdot}65 \text{ million.}$$

Alternatively, if the company could get a perfect information (or forecast) of demand (high or moderate), it should consider paying upto Rs. 0·65 million for an information.

The expected value of perfect information in business helps in getting an absolute upper bound on the amount that should be spent to get additional information on which a given decision is based.

(d) The opportunity loss values are shown below :

Table 18·6 Conditional Opportunity Loss Table

States of Nature	Probability	Conditional Profit (Rs. million)		Loss (Rs. million)	
		Courses of Action		**Courses of Action**	
O_j	$P(O_j)$	S_1	S_2	S_1	S_2
O_1	0·35	4	1	0	3
O_2	0·65	– 1	0	1	0

The coditional opportunity loss vlaues may be explained as : If outcome O_1 occurred, then the maximum profit of Rs. 4 million would be achieved by selecting course of action S_1. Thus, the choice of S_1 would result in zero opportunity loss, as it is the best decision if outcome O_1 occurs. If course of action S_2 were chosen with a payoff of one million, then this would result in a opportunity loss of 4 – 1 = Rs. 3 million. If the outcome O_2 occurred, then the best course of action would be S_2 with zero loss. Thus, no opportunity loss would be associated with the choice of S_2. But, if S_1 were chosen, then the opportunity loss would be $0 - (-1) =$ Rs. 1 million. That is, the company would have Rs. 1 million worse off in that situation, if it had chosen course of action S_1.

Using the given forecast of probabilities associated with each state of nature $P(O_1) = 0{\cdot}35$ and $P(O_2) = 0{\cdot}65$, the expected opportunity losses for the two courses of action are :

$$EOL(S_1) = 0{\cdot}35(0) + 0{\cdot}65(1) = \text{Rs. } 0{\cdot}65 \text{ million}$$

$$EOL(S_2) = 0{\cdot}35(3) + 0{\cdot}65(0) = \text{Rs. } 1{\cdot}05 \text{ million}$$

Since decision-maker seeks to minimize the expected opportunity loss, he must select course of action S_1 to produce the smallest expected opportunity loss.

Example 4. *A producer of boats has estimated the following distribution of demand for a particular kind of boat :*

No. demanded :	0	1	2	3	4	5	6
Probability :	0·14	0·27	0·27	0·18	0·09	0·04	0·01

Each boat cost him Rs. 7,000 and he sells them for Rs. 10,000 each. Any boat that are left unsold at the end of the season must be disposed off for Rs. 6000 each. How many boats should be in stock so as to miximize his expected profit ? **[Sardar Patel (M.B.A.) 96]**

Solution. Here we are given that

Cost of boat = Rs. 7,000; Selling price = Rs. 10,000.

∴ Profit if it is sold = Rs. 10,000 – Rs. 7,000 = Rs. 3,000.

If it is unsold, disposable price = Rs. 6,000.

Using the above information, the various conditional profit (payoff) values for each act-event combination are given by :

Conditional profit = MP x Boats sold – ML x Boats unsold

= (10,000 – 7,000) x Boats sold – (7000 – 6000) x Boats unsold

$$= \begin{cases} 3,000\,S & = 3,000\,S; \quad \text{if } P \le S \\ 3,000\,S + 1,000\,(S - P). & = 4,000\,S - 1,000\,P \quad \text{if } P > S \end{cases}$$

where P is the number of boats produced and S is the number of boats sold in past :

The resulting conditional payoff are computed in the following table.

Table 18.7 : Conditional Profit Values (Payoffs)

Event (Sales per season)	Prob.	Conditional Payoff Act (Production per week)						
		0	1	2	3	4	5	6
0	0·14	0	– 1000	– 2000	– 3000	– 4000	– 5000	– 6000
1	0·27	0	3000	2000	1000	0	– 1000	– 2000
2	0·27	0	3000	6000	5000	4000	3000	2000
3	0·18	0	3000	6000	9000	8000	7000	6000
4	0·09	0	3000	6000	9000	12000	11000	10000
5	0·04	0	3000	6000	9000	12000	15000	14000
6	0·01	0	3000	6000	9000	12000	15000	18000

The calculations for expected payoffs and EMV for each act are shown in the following table :

Table 18.8 : Computation Of Expected Payoff and Emv

Event (demand)	Prob.	Expected payoff Act (Production per week)						
		0	1	2	3	4	5	6
	(1)	(1) × (2)	(1) × (3)	(1) × (4)	(1) × (5)	(1) × (6)	(1) × (7)	(1) × (8)
0	0·14	0	– 140	– 280	– 420	– 560	– 700	– 840
1	0·27	0	810	540	270	0	– 270	– 540
2	0·27	0	810	1620	1350	1080	810	540
3	0·18	0	540	1080	1620	1440	1260	1080
4	0·09	0	270	540	810	1080	990	900
5	0·04	0	120	240	360	480	600	560
6	0·01	0	300	60	90	120	150	180
EMV (Rs.) :			2440	3800	**4080***	3640	2840	1880

From the above table, we find that the maximum expected profit is 4080 which occurs when stock is 3 boats. Hence the order of 3 boats per seaon will maximize his expected profit.

Example 5. *XYZ company manufactures parts for passenger cars and sells them in lots of 10,000 parts each. The company has a policy of inspecting each lot before it is actually shipped to the retailer. Five inspection categories established for quality control represent the percentage of defective items contained in each lot.*

These are given in the following table. The daily inspection chart for past 100 inspections shows the following rating or breakdown inspection :

The management is considering two possible courses of action :

(i) Shut-down the entire plant operations and thoroughly inspect each machine.

Rating	Proportion of defective Items	Frequency
Excellent (*A*)	0·02	25
Good (*B*)	0·05	30
Acceptable (*C*)	0·10	20
Fair (*D*)	0·15	20
Poor (*E*)	0·20	5
		Total = 100

(ii) Continue production as it now exists but offer the customer a refund for defective items that are discovered and subsequently returned. The first alternative will cost Rs. 600 while the second alternative will cost the company Re. 1 for each defective item that is returned.

What is the optimum decision for the company ? Find the EVPI. **[M.S. Baroda (M.B.A.,) 95]**

Solution.

Table 18.9 : Computation of Inspection and Refund Cost

Rating	Defective rate	Probability	Cost		Opportunity loss	
			Inspect	Refund	Inspect	Refund
	(1)	(2)	(3)	(4)	(5)	(6)
A	0·02	0·25	600	200*	400	0
B	0·05	0·30	600	500	100	0
C	0·10	0·20	600	1,000	0	400
D	0·15	0·20	600	1,500	0	900
E	0·20	0·05	600	2,000	0	1,400
		1·00	**600***	670	**170***	240

The cost of refund is calculated as follows :

For lot A : 10,000 × 0·02 × 1·00 = Rs. 200. Similarly, the cost of refund for other lots is calculated.

Expected cost of inspective is :

600 × 0·25 + 600 × 0·30 + ... + 600 × 0·05 = Rs. 600

Expected cost of refund is :

200 × 0·25 + 500 × 0·30 + ... + 2000 × 0·05 = Rs. 670

Since the cost of refund is more than the cost of inspection, the plant should be shut down for inspection.

Further EVPI = EOL of inspection = Rs. 170.

Example 6. *A departmental store with a bakery section is faced with the problem of how many cakes to buy in order to meet the day's demand. The departmental store prefers not to sell day old cakes in competition, leftover cakes are, therefore, a complete loss. On the other hand, if a customer desires a cake and all of them have been sold, the customer will buy elsewhere and the sales will be lost. The store has, therefore, collected information on the past sales based on selected 100-day period as shown in the table below :*

Sales per day	:	15	16	17	18
Number of days	:	20	40	30	10
Probability	:	0·20	0·40	0·30	0·10

Construct the conditional profit and the opportunity loss tables. What is the optimal number of cakes that should be bought each day. A cake costs Rs. 2 and sells for Rs. 2·50.

Solution. Marginal profit (MP) = Selling price – cost = 2·50 – 2·00 = Rs. 0·50

Marginal loss (ML) = Loss on unsold cake = Rs. 2·00

Therefore, conditional profit value = MP × cakes sold – ML × cakes not sold

= 0·50 × cakes sold – 2·00 × cakes not sold

Let O_j (j = 1, 2, 3, 4) be the possible states of nature (*i.e.*, daily demand of cakes) and S_i (i = 1, 2, 3, 4) be the all possible courses of action concerning stock policy of cakes.

The conditional profit values (payoffs) for each pair of courses of action and state of nature combinations are given in Table 18·10.

Table 18·10 Conditional Profit Table

States of nature (Sales per day)	Probability	Conditional profit (Rs.)			
O_j	$P(O_j)$	S_1 (stock 15)	S_2 (stock 15)	S_3 (stock 16)	S_4 (stock 18)
15	0·20	7·50	5·50	3·50	1·50
16	0·40	7·50	8·00	6·00	4·00
17	0·30	7·50	8·00	8·50	6·50
18	0·10	7·50	8·00	8·50	9·00

Using the conditional profit values as given in Table 18·10 and the probabilities of states of nature, the expected monetary value can be obtained for each course of action as shown below :

Table 18·11 Expected Monetary Value (EMV)

States of Nature	Probability	Conditional Profit (Rs.)				Expected Profit (Rs.)			
O_j	$P(O_j)$	S_1	S_2	S_3	S_4	S_1	S_2	S_3	S_4
15	0·20	7·50	5·50	3·50	1·50	1·50	1·10	0·70	0·30
16	0·40	7·50	8·00	6·00	4·00	3·00	3·20	2·40	1·60
17	0·30	7·50	8·00	8·50	6·50	2·25	2·40	2·55	1·95
18	0·10	7·50	8·00	8·50	9·00	0·75	0·80	0·85	0·90
					EMV	7·50	7·50	6·50	4·75

The maximum value of *EMV* is corresponding to two courses of actions S_1 (stock 15) and S_2 (stock 16). Hence, according to *EMV* criterion, the store can buy either 15 or 16 cakes, which gives the maximum *EMV*, *i.e.*

$$EMV^* = EMV(S_1) = EMV(S_2) = \text{Rs. } 7{\cdot}50$$

Example 7. *A modern home appliances dealer finds that the cost of holding a mini cooking range is stock for a month is Rs. 200 (insurance, minor deterioration, interest on borrowed capital, etc.) Customer who cannot obtain a cooking range immediately tends to go to other dealers and he estimates that for every customer who cannot get immediate delivery, he loses an average of Rs. 500. The probabilities of a demand of 0, 1, 2, 3, 4, 5 mini cooking ranges in a month are* 0·05, 0·10, 0·20, 0·30, 0·20, 0·15 *respectively. Determine the optimum stock level of cooking rangers. Also find EVPI.* **[Delhi (M.B.A.) 96]**

Solution. The cost function = $\begin{cases} \text{Rs. } 500\,(D-S), & \text{if } D \geq S \\ \text{Rs. } 200\,(S-D), & \text{if } D < S \end{cases}$

where S is number of units purchased and D is the number of units demanded.

Table 18.12 : Computation Of Expected Cost

Event (Demand)	Prob.	Conditional cost (Rs.) Act						Expected cost (Rs.) Act					
		0	1	2	3	4	5	0	1	2	3	4	5
0	0·05	0	200	400	600	800	1000	0	10	20	30	40	50
1	0·10	500	0	200	400	600	800	50	0	20	40	60	80
2	0·20	1000	500	0	200	400	600	200	100	0	40	80	120
3	0·30	1500	1000	500	0	200	400	450	300	150	0	60	120
4	0·20	2000	1500	1000	500	0	200	400	300	200	100	0	40
5	0·15	2500	2000	1500	1000	500	0	375	300	225	150	7	50
Expected cost								1475	1010	615	360	**315***	410

Since expected cost is minimum if 4 cooking ranges are stocked each month, the optimum act is to stock 4 cooking ranges.

Example 8. *A TV dealer find that the cost of holding a TV in stock for a week is Rs. 50. Customers who cannot obtain new TVs immediately tend to go to other dealers and he estimates that for every customer who cannot get immediate delivery he loses an average of Rs. 200. For one particular model of TV, the probabilities of a demand of 0, 1, 2, 3, 4 and 5 TVs in a week are 0·05, 0·10, 0·20, 0·30, 0·20 and 0·15 respectively.*

(i) How many televisions per week should the dealer order ? Assume that there is no time lag between ordering and delivery.

(ii) Compute EVPI.

(iii) The dealer is thinking of spending on a small market survey to obtain additional information regarding the demand levels. How much should be willing to spend on such a survey ? **[Delhi (M.B.A.) Nov. 97]**

Solution. If D denotes the demand and S the number of televisions stored (ordered), then the conditional cost values are computed as follows :

$$\text{Cost function} = \begin{cases} 50\,S + 200\,(D - S\,), \text{ when } D \geq S \\ 50\,D + 50\,(S - D), \text{ when } D < S \end{cases}$$

Table 18.13 : Computation Of Expected Cost

Event (Demand)	Probability	Conditional Cost (Rs.) Act (Stock)					
		0	1	2	3	4	5
0	0·05	0	50	100	150	200	250
1	0·10	200	50	100	150	200	250
2	0·20	400	250	100	150	200	250
3	0·30	600	450	300	150	200	250
4	0·20	800	650	500	350	200	250
5	0·15	1000	850	700	550	400	250
Expected cost		590	450	330	250	230*	250

Table 18.14 : Computation of Ecpi

Event (Demand) (1)	Probability (2)	Minimum cost under perfect information (3)	Cost under expected perfect information (4) = (2) × (3)
0	0·05	0	0
1	0·10	50	5
2	0·20	100	20
3	0·30	150	45
4	0·20	200	40
5	0·15	250	37·5
Total			ECPI = 147·5

EVPI = EC* – ECPI = 230 – 147·5 = Rs. 82·5.

(implying profit of Rs. 200 per TV for the shortage cost of Rs. 200). The pay-off table is compiled below :

Table 18.15 : Computation of Emv ,

Event (Demand)	Probability	Conditional pay-off Act (Stock)					
		0	1	2	3	4	5
0	0·05	0	– 50	– 100	– 150	– 200	– 250
1	0·10	0	150	100	50	0	– 50
2	0·20	0	150	300	250	200	150
3	0·30	0	150	300	450	400	350
4	0·20	0	150	300	450	600	550
5	0·15	0	150	300	450	600	750
EMV		0	140	260	340	360*	340

Table 18.16 : Computation of Eppi

Event Demand (1)	Probability (2)	Pay-off under perfect information (3)	Expected pay-off under perfect information (4) = (2) × (3)
0	0·05	0	0
1	0·10	150	15
2	0·20	300	60
3	0·30	450	135
4	0·20	600	120
5	0·15	750	112·5
Total			442·5

The expected value of perfect information is given by :

$$\text{EVPI} = \text{EPPI} - \text{EMV}^* = 442{\cdot}5 - 360 = \text{Rs. } 82{\cdot}5.$$

(iii) On the basis of the given data, the dealer should not be willing to spend more than Rs. 82·5 for the market survey.

Example 9. *A certain piece of equipment is to be purchased for a construction project at a remote location. This equipment contains an expensive part which is subject to random failure. Spares of this part can be purchased at the same time the equipment is purchased. Their unit cost is Rs. 1,500 and they have no scrap value. If the part fails on the job and no spare is available, the part will have to be manufactured on a special order basis. If this is required, the total cost, including down time of the equipment is estimated as Rs. 9,000 for each such occurrence. Based on previous experience with similar parts, the following probability estimates of the number of failures expected over the duration of the project are provided as given below :*

Failure :	0	1	2	3 and above
Probability :	0·80	0·15	0·05	0·00

(a) Determine optimal EMV and optimal number of spares to purchase initially.*

(b) Based on opportunity losses, determine the optimal course of action and optimal value of EOL.

(c) Determine expected profit with perfect information and expected value of perfect information.

Solution. (a) Let O_1 (no failure), O_2 (1 failure) and O_3 (2 failures) be the possible states of nature or outcomes (i.e., number of parts failures or number of spares required). Similarly, let S_1 (no spare purchased), S_2 (1 spare purchased), and S_3 (2 spare purchased) be possible courses of action or strategies.

Table 18·17 Conditional Costs

States of nature (spare required)	Courses of action (number of spare purchased)	Purchase cost (Rs.)	Emergency cost (Rs.)	Total cost (Rs.)
0	0	0	0	0
0	1	1,500	0	1,500
0	2	3,000	0	3,000
1	0	0	9,000	9,000
1	1	1,500	0	1,500
1	2	3,000	0	3,000
2	0	0	18,000	18,000
2	1	1,500	9,000	10,500
2	2	3,000	0	3,000

Using conditional costs and the probabilities of states of nature, the expected monetary value can be determined for each of three states of nature as shown below :

Table 18·18. Expected Monetary Value (EMV)

State of nature	Probability	Conditional costs (Rs.) Courses of action			Weighted costs (Rs.) Courses of action		
O_j	$P(O_j)$	S_1	S_2	S_3	S_1	S_2	S_3
O_1	0·80	0	1,500	3,000	0·80 (0) = 0	1,200	2,400
O_2	0·15	9,000	1,500	3,000	0·15 (9,000) = 1,350	225	450
O_3	0·05	18,000	10,500	3,000	0·05 (18,000) = 900	525	150
				EMV	2,250	1,950	3,000

To minimize the cost, course of action S_2 should be selected. If *EMV* is expressed in terms of profit, then

$$EMV^* = EMV(S_2) = -\text{Rs. } 1{,}950$$

Hence, the optimal number of spares to be purchased initially should be one.

(b) To determine *EOL*, we first determine *COL*. The calculations for conditional opportunity loss (*COL*) are given below :

Table 18·19

States of nature	Conditional cost (Rs.)			Conditional opportunity loss (Rs.)		
O_j	S_1	S_2	S_3	S_1	S_2	S_3
O_1	0	1,500	3,000	0	1,500	3,000
O_2	9,000	1,500	3,000	7,500	0	1,500
O_3	18,000	10,500	3,000	15,000	7,500	0

(Since we are dealing with the conditional cost rather than conditional profit, lower value for each state of nature shall be considered for determining opportunity losses).

The calculations for expected opportunity loss are shown in table below.

Table 18·20. Expected Opportunity Loss (EOL)

States of nature	Probability	Conditional opportunity loss (Rs.)			Weighted opportunity loss (Rs.)		
O_j	$P(O_j)$	S_1	S_2	S_3	S_1	S_2	S_3
O_1	0·80	0	1,500	3,000	0·80 (0) = 0	1,200	2,400
O_2	0·15	7,500	0	1,500	0·15 (7,500) = 1,125	0	225
O_3	0·05	15,000	7,500	0	0·05 (15,000) = 750	375	0
					1,875	1,575	2,625

Since $EOL^* = EOL(S_2) =$ Rs. 1,575, it is advisable to adopt course of action S_2 and to purchase one spare.

(c) The expected profit with perfect information (*EPPI*) can be determined by choosing optimal course of action for each state of nature, multiplying its conditional value by the probability $P(O_j)$, $j = 1, 2, 3$ and then summing these products. The *EPPI* calculations are given in Table 18·21.

Table 18·21.

State of nature O_j	Probability $P(O_j)$	Optimal course of action	Cost of optimal course of action (Rs.)	
			Conditional cost	Weighted opportunity loss
O_1	0·80	S_1	0	0·80 (0) = 0
O_2	0·15	S_2	1,500	0·15 (1,500) = 225
O_3	0·05	S_3	3,000	0·05 (3,000) = 150
				Total 375

Since we are dealing with the expected cost, $EPPI = -$ Rs. 375

Now the expected vlaue of perfect information is :

$$EVPI = EPPI - EMV^* = -375 - (-1950) = \text{Rs. } 1{,}575$$

Here it can be observed that $EVPI = EOL^* =$ Rs. 1575.

Example 10. *Under an employment promotion programming, it is proposed to allow sale of newspapers on the buses during off-peak hours. The vendor can purchase the newspaper at a special concessional rate of 25 paise per copy against the selling price of 40 paise. Unsold copies are, however, a dead loss. A vendor has estimated the following probability distribution for the number of copies demanded,*

Number of copies demanded	:	15	16	17	18	19	20
Probability	:	0·04	0·19	0·33.	0·26	0·11	0·07

How many copies should he order so that his expected profit will be maximum ?

Solution. The vendor does not purchase less than 15 copies or more than 20 copies. Let n be the number of copies of newspaper demanded. The vendor would lose 25 paise on each copy in case demand is less than n. Otherwise, if the demand is more than or equal to n, then he would gain 15 paise on each newspaper copy.

The incremental profit (expected profit – expected loss) for each value of n is given in Table 18·22.

Table 18·22. Expected Profit

Demand (n)	Probability less than n)	Probability more than n)	Expected incremental profit (Rs.)	Total profit
15	0·00	1·00	0 (0) + 1 (0·15) = 0·15	0·15 × 15 = 2·25
16	0·04	0·96	0·04 (– 0·25) + 0·96 (0·15) = 0·13	2·38
17	0·23	0·77	0·23 (– 0·25) + 0·77 (0·15) = 0·06	2·44
18	0·56	0·44	0·56 (– 0·25) + 0·44 (0·15) = – 0·07	2·37
19	0·82	0·18	0·82 (– 0·25) + 0·18 (0·15) = – 0·18	2·19
20	0·93	0·07	0·93 (– 0·25) + 0·07 (0·15) = – 0·22	1·97

The expected profit is maximum at $n = 17$, and hence the vendor should order 17 copies of newspaper.

Example 11. *A company manufactures goods for a market in which the technology of the products is changing rapidly. The research and development department has produced a new product which appears to have potential for commercial exploitation. Further, Rs. 60,000 is required for development testing.*

The company has 100 customers and each customer might purchase, at the most, one unit of the product. Market research suggests a selling price of Rs. 6,000 for each unit with total variable costs of manufacture and selling estimate are at Rs. 2,000 for each unit.

As a result of previous experiment of this type of market, it has been possible to derive a probability distribution relating to the proportions of customers who will buy the product as follows :

Proportion of customers :	0·04	0·08	0·12	0·16	0·20
Probability :	0·10	0·10	0·20	0·40	0·20

Determine the expected opportunity losses, given no other information than that stated above, and to state whether, or not, the company should develop the product.

Solution. Let p be proportion of customer purchasing the new product. Then conditional profit is

$$= (6{,}000 - 2{,}000) \times 100p - 60{,}000 = \text{Rs. } (4{,}00{,}000\,p - 60{,}000)$$

Also O_j $(j = 1, 2, 3, 4, 5)$ be the possible states of nature (*i.e.*, proportion of customers who will buy the new product) and S_1 (to develop the product) and S_2 (not to develop the product) be the two courses of action.

The conditional profit values (payoffs) for each pair of O_j $(j = 1, 2, 3, 4, 5)$ and S_i $(i = 1, 2)$ is given below in Table 18·23.

Table 18·23. Payoff Table : Conditional Profit (Rs.)

States of nature (Proportion of customers) O_j	Courses of action	
	S_1 (Develop the product)	S_2 (Not to develop the product)
0·04	– 44,000	0
0·08	– 28,000	0
0·12	– 12,000	0
0·16	4,000	0
0·20	20,000	0

The opportunity loss value are shown in table 18·24

Table 18·24. Opportunity Loss

States of nature	Probability	Conditional profit (Rs.)		Opportunity loss (Rs.)	
		Courses of action		Courses of action	
O_j	$P(O_j)$	S_1	S_2	S_1	S_2
0·04	0·1	– 44,000		44,000	0
0·08	0·1	– 28,000	0	28,000	0
0·12	0·2	– 12,000	0	12,000	0
0·16	0·4	4,000	0	0	4,000
0·20	0·2	20,000	0	0	20,000

Using he given probabilities associated with each state of nature, the expected opportunity losses for the two courses of action are :

$$EOL\,(S_1) \;=\; 0{\cdot}1\;(44{,}000) \;+\; 0{\cdot}1\;(28{,}000) \;+\; 0{\cdot}2\;(12{,}000) \;+\; 0{\cdot}4\;(0) + 0{\cdot}2\;(0) \;=\; \text{Rs. } 9{,}600$$

$$EOL\,(S_2) \;=\; 0{\cdot}1\;(0) \;+\; 0{\cdot}1\;(0) \;+\; 0{\cdot}2\;(0) \;+\; 0{\cdot}4\;(4{,}000) \;+\; 0{\cdot}2\;(20{,}000) \;=\; \text{Rs. } 5{,}600$$

Since the company has to minimize the expected opportunity loss, it chooses the course of action S_2 , (not to develop the product), as it gives the minimum *EOL*.

Example 12. *The manager of a Flower-stop promises its customers delivery within four hours on all flower orders. All flowers are purchased on the previous day and delivered to Parker by* 8·00 *a.m., the next morning. The daily demand for roses is as follows :*

Dozens of roses :	70	80	90	100
Probability :	01	0·2	0·4	0·3

The manager purchases roses for Rs. 10 per dozen and sells them for Rs. 30. All unsold roses are donated to a local hospital. How many dozens of rozes should Parker order each evening to maximize its profit. What is the optimum expected profit ? **[Delhi (MBA) 2002, 86]**

Solution. The number of roses (in dozens) purchased per day is under the control of decision maker. This is considered as course of action or 'decision choice'. But the daily demand of the flowers is uncertain and only its probability is known. Therefore, it is considered as a 'state of nature' or event. It is obvious from the given data that the flower shop must not purchase less than 7 or more than 10 dozens of roses per day. Also, each dozen of roses sold within a day gives a profit of Rs. (30–10) = Rs. 20. Otherwise, it will be a loss of Rs. 10.

Therefore,

$$MP = \text{Marginal Profit} = \text{Selling price} - \text{Cost} = 30 - 10 = \text{Rs. } 20$$

$$ML = \text{Marginal Loss} = \text{Loss on unsold roses} = \text{Rs. } 10.$$

Now the given information can be used to find the various conditional profit (pay off) values for each combination of decision choice event by the formula

$$\text{Conditional Profit} = MP \times \text{roses sold} - ML \times \text{roses not sold}$$

$$= \begin{cases} 20D & \text{if } D \ge S \\ 20D - 10\,(S - D) = 30D - 10S, & \text{if } D < S \end{cases}$$

where *D* stands for the number of roses sold within a day and 5 the number of roses stocked. The resulting conditional profit values and corresponding expected pay offs are calculated as given in the table below :

Conditional Profit Values (Pay-offs)

States of Nature (Demand/day)	Probability	Conditional Profit (Rs.) due to courses of action (Purchase per day)				Expected Pay-off (Rs.) due to Courses of Action (Purchase per day)			
	(1)	70 (2)	80 (3)	90 (4)	100 (5)	70 (1) × (2)	80 (1) × (3)	90 (1) × (4)	100 (1) × (5)
70	0·1	140	130	120	110	14	13	12	11
80	0·2	140	160	150	140	28	32	30	28
90	0·4	140	160	180	170	56	64	72	68
100	0·3	140	160	180	200	42	48	54	60
EMV (Expected Owners Value)						140	157	168	167

Example 14. *A steel manufacturing company is concerned with the possibility of a strike. It will cost an extra Rs. 10,000 to acquire an adequate stockpile. If there is a strike and the company has not stockpiled, then management estimates an additional expense of Rs. 50,000 in lost sales, late order charges and so forth. Should the company stockpile or not ?*

Solution. With the help of given data, conditional cost table can be obtained as shown in Table 18·29.

Table 18·29

State of nature	Conditional costs (Rs.)	
	Courses of action	
	S_1 (stockpile)	S_2 (Do not stockpile)
O_1 (Strike)	10,000	50,000
O_2 (No Strike)	10,000	0

Maximax Criterion. In a cost problem, the maximax criterion really means the "minimum". Thus find the minimum of the minimum costs of all courses of action. Here, the minimum cost of S_1 is Rs. 10,000 and that of S_2 is 0. Therefore, the company would select course of action S_2 (not to stockpile), with a cost of 0.

Minimax Criterion. The maximum cost for S_1 is Rs. 10,000 and for S_2 is Rs. 50,000. The company should stockpile, at a cost of Rs. 10,000, since this is preferable to the maximum possible cost of Rs.. 50,000 if it does not stockpile.

Equal Likelihood (Laplace Criterion). If both outcomes were equally likely, then the expected cost of course of action S_1 would be Rs. 10,000 and that of S_2 would be Rs. 25,000. The company would select to stockpile if it relies that there was a equal (50%) chance of strike.

Criterion of Regret. The regret table for the problem is shown below :

Table 18·30

States of nature	Regret values (Rs.)	
	Courses of action	
	S_1	S_2
O_1	0	40,000
O_2	10,000	0

The maximum regret for S_1 is Rs. 10,000 and that of S_2 is Rs. 40,000. The company would minimize the maximum regret by stockpiling, with a maximum regret of Rs. 10,000.

Example 15. *A manufacturer makes a product, of which the principal ingredient is a chemical X. At the moment, the manufacturer spends Rs. 1,000 per year on supply of X, but there is a possibility that the price may soon increase to four times its present figure because of a worldwide shortage of the chemical. There is another chemical Y, which the manufacturer could use in conjunction with a third chemical Z in order to give the same effect as chemical X. Chemicals Y and Z would together cost the manufacturer Rs. 3,000 per year, but their prices are unlikely to rise what action should the manufacturer take ? Apply the maximin and minimax criteria for decision making and give two sets of solutions. If the coefficient of optimism is* 0·4, *find the course of action that minimizes the cost.* **[I.C.W.A. Dec. 1988]**

Solution. The data of the problem is given in the following table (negative numebrs represent profit).

State of Nature	Courses of Action	
	S_1 (Use Y and Z)	S_2 (Use X)
N_1 (increase in price of X)	– 3,000	– 4,000
N_2 (no increase in price of X)	–3,000	– 1,000

(i) Maximin Critenion

States of Nature	Courses of Action	
	S_1 (Use Y and Z)	S_2 (Use X)
N_1	– 3,000	– 4,000
N_2	– 3,000	– 1,000
Column minimum	– 3,000 maximin	– 4,000

Here, maximum of column minima =– 3,000. Hence the manufacturer should adopt action S_1.

(ii) Minimax (or Opportunity loss) Criterion.

State of Nature	Courses of Action	
	S_1	S_2
N_1	– 3,000 – (– 3,000) = 0	– 3000 – (– 4,000) = 1,000
N_2	– 1,000 – (– 3,000) = 2,000	– 1,000 – (– 1,000) = 0
Maximum opportunity	2,000	1,000 minimax

Hence manufacturer should adopt the minimum opportunity loss course of action S_2.

(iii) Hurwicz Criterion : Since the coefficient of optimism is given to be 0·4, the coefficient of pessimism will be 1 – 0·4 = 0·6. Therefore, select course of action that optimizes (maximum for profit and minimum for cost) the pay off value according to Hurwicz).

$H = \alpha$ (best profit) + $(1 - \alpha)$ (worst pay off)

= α (maximum in column) + $(1 - \alpha)$ (minimum in column)

Course of action	Best pay off	Worst pay off	H
S_1	– 3,000	– 3, 000	– 3,000
S_2	– 1,000	– 4,000	– 2,800

Since course of action S_2 has the least cost maximum profit = 0·4 (1,000) + 0·6 (4,000) = 2,800, the manufacturer should adopt it.

Example 16. *A food product company is contemplating the introduction of a revolutionary new product with new pakasing or replace the existing product at much higher price (S_1) or a moderate change in the composition of the existing product with a new packaging at a small increase in price (S_2) or a small change in the composition of the existing product except the word 'new' with a neglisible increase in price (S_1). The three possible states of nature or events are : (i) high increase in sales (N_1) (ii) no change in sales (N_2) and (iii) decrease in sales (N_3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three events (expected sales). This is represented in the following table :*

Strategies	States of Nature		
	N_1	N_2	N_3
S_1	7,00,000	3,00,000	1,50,000
S_2	5,00,000	4,50,000	0
S_3	3,00,000	3,00,000	3,00,000

Which strategy should the concerned executive choose on the basis of—

(i) Maximin criterion (ii) Maximax criterion

(iii) Minimax regret criterion (iv) Laplace criterion.

Solution. We may write the pay off matrix as given below :

(i) Maximin Criterion

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column Minimum	1,50,000	0	**3,00,000** (maximin)

Since the maximum of column minima is 3,00,000, the company should accept strategy S_3.

(ii) Maximax Criterion

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column Maximum	**7,00,000** (maximax)	5,00,000	3,00,000

Since the maximum of column maxima is 7,00,000 the company should accept the strategy S_1.

(iii) Minimax Regret Criterion : The *opportunity loss* can be represented as shown in table below :

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	0	2,00,000	4,00,000
N_2	1,50,000	0	1,50,000
N_3	5,50,000	3,00,000	0
Column maximum	5,50,00,000 (Minimax regret)	3,00,000	4,00,000

Therefore, the company should accept the minimum opportunity loss stratesy, S_1.

Example 17. *The firm manufactures three types of products. The fixed and variable costs are given below :*

		Fixed cost (Rs.)	Variable cost/unit (Rs.)
	A	25,000	12
Product	B	35,000	9
	C	53,000	7

The likely demand (units) of the products is given below :

Poor demand = 3000; *moderate Demand* = 7,000; *High Demand; 11,000.*

If the sale price of each type of product is Rs. 25, then prepare the pay off matrix.

Solution. Let the poor, moderate and high demand be D_1, D_2, D_3 respectively. Then *Payoff* is given by (*Sales revenue– Cost*). Therefore the computation for payoff for each pair of alternative demands (course of actions) and the types of product (states of nature are given below (in thousands) :

$$D_1A = 14, D_2A = 66, D_3A = 118$$

$$D_1B = 13, D_2B = 77, D_3B = 141$$

$$D_1C = 1, D_2C = 73, D_3C = 145.$$

Thus the payoff values are shown in table below :

Types of Product	Alternative Demand		
	D_1	D_2	D_3
A	14	66	118
B	13	77	141
C	1	73	145

Example 18. *A retailer purchases cherries every mornng at Rs. 50 a case and sells them for Rs. 80 a case. Any case remaining unsold at the end of the day can be disposed of next day at a salvage value of Rs. 20 per case (thereafter they have no value). Past sales have ranged from 15 to 18 cases per day. The following is the record of sales for the past 120 days :*

Cases sold :	15	16	17	18
Number of days :	12	24	48	36

Find how many cases the retailer should perchase per day to maximize the profit.

[Ajmer (MBA) 1988, Delhi (M. Com) 85]

Solution. Suppose N_i (i= 1, 2, 3, 4) denotes the possible sates of nature (daily likely demand) and S_j (j = 1, 2, 3, 4) the all possible cources of action (number of cases of cherries to be purchased).

MP = Marginal Profit = Selling price – Cost = Rs. (80 – 50) = Rs. 30.

ML = Marginal Loss = Loss due to unsold cases – Rs. (50 – 20) = Rs. 30.

The conditional profit (payoff) value for each act-event combination are given by the formula :

Conditional profit = MP × Sold cases – ML × Unsold cases

$$= (80 - 50) \text{ Sold cases} - (50 - 20) \text{ Unsold cases}$$

$$= \begin{cases} 30S & \text{if } S \geq N \\ (80 - 50)\, S - 30\,(N - S) = 60S - 30N & \text{if } S < N \end{cases}$$

The resulting conditional profit values and corresponding expected payoffs are given below in the table :

Conditional Profit Values (Pay-offs)

State of Nature (Demand/weak)	Probability	Conditional Profit (Rs.) due to courses of Action (Purchase/day)				Expected Pay off (Rs.) due to courses of Action (Purchase per day)			
		15	16	17	18	15	16	17	18
	(1)	(2)	(3)	(4)	(5)	(1) × (2)	(1) × (3)	(1) × (4)	(1) × (5)
15	0·1	450	420	390	360	45	42	39	36
16	0·2	450	480	450	420	90	96	90	84
17	0·4	450	480	510	480	180	192	204	192
18	0·3	450	480	510	540	135	144	153	162
EMV (Expected Monetary Value)						450	474	486	474

Since maximum EMV of Rs. 486 is corresponding to course of action 17, the retailer must purchase 17 cases of cherries every morning.

18.7. DECISION-MAKING UNDER CONFLICT

Decision-making under conflict shall be discussed in the chapter on ***"The Theory of Games,"*** :

18.8. DECISION TREE ANALYSIS

Decision-making problems discussed so far are referred to as single stage decision problems. Because the states of nature, payoffs, courses of action and probability distribution are not subject to change under the assumption that no new information is sought and time also does not change any basic component of decision-making environment. But situations may arise when decision-maker needs to revise his previous decisions and make a sequence of other decisions. Thus the problem becomes, a *multistage decision problem* because the outcome of one decision affects the further decisions. For example, in the process of marketing, a new type of product, the first decision is often test the marketing, and the action choice might be between intensive and gradual testing. Given various possible outcomes (favourable, fair or poor), we may require to decide amongst re-designing the product, an energetic advertising campaign or complete withdrawal of the product and so on. Given the decision, there will be an outcome leading to another decision, and so on.

Time and uncertainty are two important features of this kind of decision problem. Both courses of action and states of nature take some time to initiate. In this process, the total elapsed time may be very large. The effect of time on financial values can be incorporated by discounting. But, probabilities may change as the time passes, and also emerges new and more information.

The other factor is the *uncertainty*. With many outcomes or states of nature, it is difficult to have assessment of probabilities in case of only one course of action. Also, it is equally difficult to make a set of mutually consistent probabilities for the whole chain of outcomes. Despite of these difficulties decision-maker has to solve such problems.

The type of sequential problem discussed so far can best be represented by a *decision tree.* A decision tree is thus, a schematic representation of a sequential and multi-dimensional decision problems.

A decision tree is made of *nodes, branches; probability estimates,* and *payoffs.* There are two types of nodes : *decision nodes* and *chance nodes.* A *decision node* is usually indicated by a square and represents places where a decision-maker must make a decision. Each branch leading away from a node indicates one of the several possible courses of action available to the decision-maker. The chance node is indicated by a circle and represents a point at which the decision-maker will discover the response to his decision, *i.e.* different possible outcomes (states of nature, competitors actions, etc.) which can be obtained from a chosen course of action.

Branches emanating from and connecting various nodes are either decisions or states of nature. There are two types of branches : *decision branches* and *chance branches.* Each branch leading away from decision node indicates a course of action or strategy that can be selected at this decision point, which branch is leading away from a chance node indicates the state of nature of a set of chance factors. Associated probabilities are represented along side of respective chance branch. These probabilities are the likelihood that the chance outcome will assume the value assigned to the given branches. Any branch that makes the end of the decision tree (*i.e.*, it is not followed by either a decision or chance node) is called a *terminal branch.* A *terminal branch* can indicate either a course of action or a chance outcome.

The *payoff* can be positive (*i.e.*, revenue or sales) or negative (*i.e.*, costs or expenditure) and they can be associated either with decision or the chance branches.

It is possible for a decision tree to be deterministic or probabilistic and it can be further divided into a single stage (*a decision under condition of certainty*) and a multistage (*a sequence of decisions*).

The optimal sequence of decisions in a tree is determined by starting at the right hand side and "rolling backward". In operation, we have to *maximize the return* from the decision situation. At each node, an *expected return* is calculated which is also called the *position value.* If the node is a chance node, then the position value is determined as the sum of products of probabilities on the branches emanating from the chance node and their respective position values. If the node is decision, then the expected return is calculated for each of its branches and the highest return is chosen. The procedure continues until the initial node is reached. The position values corresponds to maximum expected return obtainable from the decision sequence.

Example 19. *Raman Industries Ltd., has a new product which they expect has great potential. At the moment they have two courses of action open to them :* S_1 = *To test the market,* S_2 = *To drop the product.*

If they test it, it will cost Rs. 50,000 and the response could be positive or negative with probabilities of 0·70 and 0·30 *respectively. If it is positive, they could either market it with full scale or drop the product. If they market with full scale, then the result might be low, medium, or high demand, and the respective net pay-offs would be – Rs. 100,000, Rs. 100,000 or Rs. 500,000. These outcomes have probabilities of* 0·25 , 0·55 and 0·20 *respectively. If the result of the test marketing is negative, they have decided to drop the product. If, at any point, they drop the product there is a net gain of Rs. 25,000 from the sale of scrap. All financial values have been discounted to the present.*

Draw a decision tree for the problem and indicate the most preferred decision.

Solution. The decision tree, indicating possible courses of action and outcomes is shown in Fig 18·2.

In order to analyse the tree, start working backward (roll back process) from the end branches. Nodes F , G and H are determined from node E as a result of a probabilistic outcome in the sense that having reached node E , there is no means of knowing where we will end up except that it will be one of F , G or H. The resulting payoff values for low, medium and high demand is given as Rs. 100,000; Rs. 100,000 and Rs. 500,000 respectively. The expected monetary value (EMV) for node E is therefore calculated as below :

$$EMV(E) = 0{\cdot}20 \times 500{,}000 + 0{\cdot}55 \times 100{,}000 + 0{\cdot}25\ (-\ 100{,}000) = \text{Rs. } 1{,}30{,}000$$

Chance node E and D are obtained from decision node 2. Then at node 2, we can select the node to move to. If we select to move to E , there will be a certain payment of Rs. 130,000 and if we select to move to D , there will be a certain payment of Rs. 25,000. As our principle is of maximum expected monetary value, we

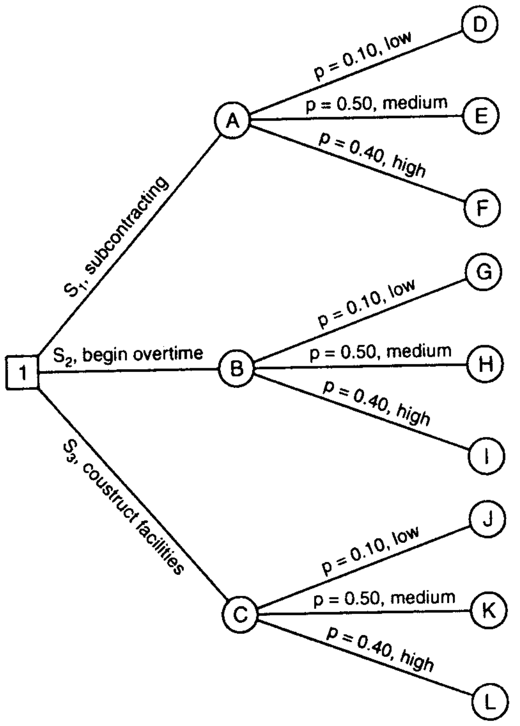


Fig. 18.2.

select the course of action. 'Test market' branch (2-*E*) is marked as the optimal course of action which is conditional upon being at the decision node 2. The value of decision node 2 is then max. (25,000, 130,000) = Rs. 130,000.

Decision node 3 leads only to chance node *C* so the value of node 3 in Rs. 25,000. The value of node *A* is now obtained by taking the *EMV* of the outcomes 'positive' and 'negative', which as we have seen, leading to sub-trees having value of Rs. 130,000 and Rs. 25,000 respectively.

$$EMA\ (A) = 0{\cdot}70 \times 130{,}000 + 0{\cdot}30 \times 25{,}000 = \text{Rs. } 98{,}500.$$

The value of branch $(1 - A)$ is 98,500 — 50,000 = Rs. 48,500 and the value of branch $(1 - B)$ is Rs. 25,000. We therefore, identify the optimal initial course of action as 'test market'. The max. (48500, 25000) = Rs. 48500, i.e., corresponding to branch $(1 - A)$, *i.e.*, 'test market'. We have now worked out a contingency plan suggesting that —

(a) The company should test the market rather than drop the product.

(b) If the 'test market' result is positive, then company should market the product.

(c) If the test market is negative, then company should drop the product.

Example 20. *A glass factory specializing in crystal is developing a substantial backlog and the firm's management is considering three courses of action : (S_1) arrange for sub-contracting, (S_2) construct new facilities. The correct choice depends largely upon future demand which may be low, medium, or high. By consensus, management ranks the respective probabilities as* 0·10 , 0·50 *and* 0·40. *A cost analysis reveals the effect upon the profits that is shown in the table below :*

Profit (Rs. '000) if demand is	Courses of action		
	S_1 (Subcontracting)	S_2 (Overtime)	S_3 (Construct facilities)
Low ($p = 0{\cdot}10$)	10	− 20	− 150
Medium ($p = 0{\cdot}50$)	50	60	20
High ($p = 0{\cdot}40$)	50	100	200

Show this decision situation is in the form of a decision tree and indicates the most preferred decision and corresponding expected value.

Solution. A decision tree representing the possible courses of action and states of nature are shown in Fig. 18·3. In order to analyse the tree, start working backward from the end branches.

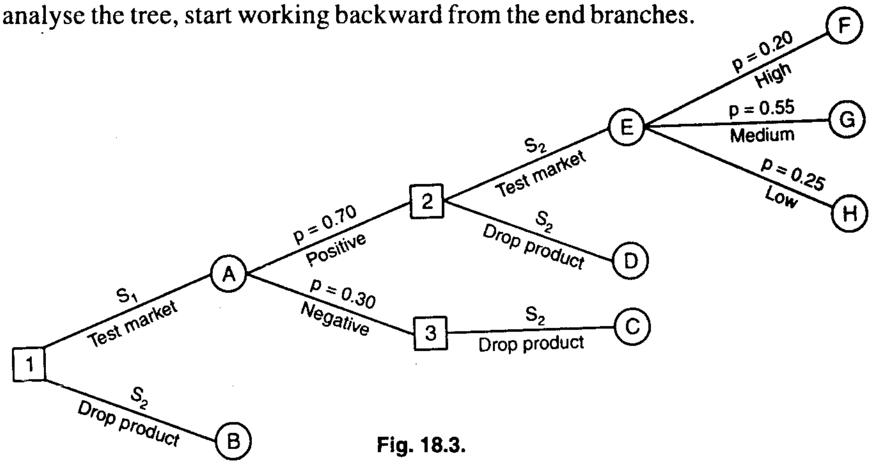


Fig. 18.3.

The most preferred decision at the decision node 1 is obtained by calculating expected vlaue of each decision branch and then selecting the path (course of action) with high value.

The expected monetary value for node A , B and C is obtained as follows :

$$EMV(A) = 0{\cdot}10 \times 10 + 0{\cdot}50 \times 50 + 0{\cdot}40 \times 50 = \text{Rs. } 46{,}000$$
$$EMV(B) = 0{\cdot}10 \times (-20) + 0{\cdot}50 \times 60 + 0{\cdot}40 \times 100 = \text{Rs. } 68{,}000$$
$$EMV(C) = 0{\cdot}10 \times (-150) + 0{\cdot}50 \times 20 + 0{\cdot}40 \times 200 = \text{Rs. } 75{,}000$$

Since node C has the highest *EMV,* the decision at node 1 will be to select the course of action S_3 , *i.e.* construct new facilities.

Example 21. *A manufacturing company has just developed new product. On the basis of past experience, a product such as this will either be successful, with an expected gross return of Rs. 100,000, or unsuccessful, with an expected gross return of Rs. 20,000. Similar products manufactured by the company have a record of being successful about 50% of the time. The production and marketing costs of the new product are expected to be Rs. 50,000.*

The company is considering whether to market this new product or to drop it. Before making its decision, however, a test marketing effort can be conducted at a cost of Rs. 10,000. Based on the past experience, test marketing results have been favourable about 70% of time. Furthermore, products favourably tested have been successful 80% of the time. However, when the test marketing result has been unfavourable. the product has only been successful 30% of the time. What course of action should the company pursue ?

Solution. Let us first define the courses of action and states of nature :

Courses of action : S_1 = test market, S_2 = market product, S_3 = drop product

States of nature : O_1 = test marketing favourable, O_2 = test marketing unfavourable,
O_3 = product successful, O_4 = product unsuccessful.

The decision tree representing different courses of action and states of nature is shown in Fig 18·4.

For terminal nodes (states of nature), the payoffs are indicating the net profit (cost is expressed as negative profit). For example, test the marketing costs Rs. 10,000 and Rs. 50,000, while successful product earns a profit of Rs. 100,000. Thus, resulting (conditional) net profit is Rs. 40,000.

Evaluation of tree begins with the terminal nodes (roll back principle). Working backward from each terminal node to the nearest node (a chance node), an expected net profit is obtained for the node depending on the probabilities associated with the branches from that node. If we adopt course of action S_2 (market) at the decision node 2 then the expected monetary value at node F is :

$$EMV(F) = 0{\cdot}80 \times (40{,}000) + 0{\cdot}20 \times (-40{,}000) = \text{Rs. } 24{,}000$$

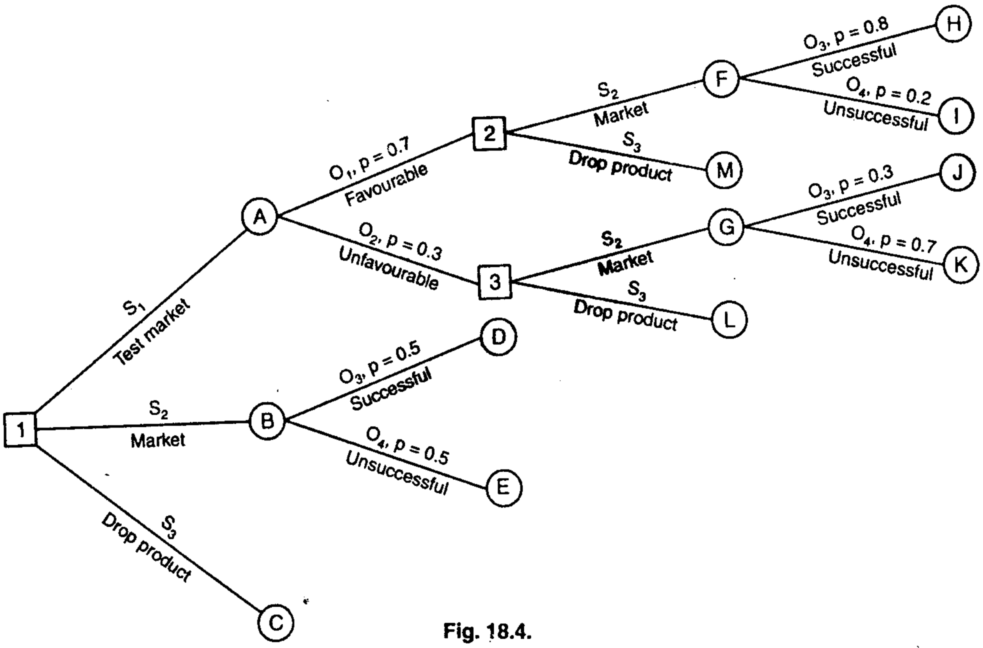


Fig. 18.4.

Chance node F and M are obtained from decision node 2, and at node 2, we can decide which node to move to. Since alternative decision, dropping the product (course of action S_3), has an expected net profit of Rs. 10,000, as per the principle of maximum EMV, the decision at node 2 would be to 'market the product'. That is, $EMV(2) = \text{Rs. } 24{,}000$.

At decision node 3, course of action S_2 (market) has an EMV as indicated below :

$$EMV(G) = 0{\cdot}3 \times (40{,}000) + 0{\cdot}7 \times (-40{,}000) = -\text{Rs. } 16{,}000$$

Again at node 3, alternative course of action S_3 (dropping the product) has an expected net profit of Rs. 10,000. Therefore, between courses of action S_2 and S_3, the preferred alternative is course of action S_3 (dropping the product), *i.e.* $EMV(3) = -\text{Rs. } 10{,}000$.

Using EMV values depending on preferred alternatives found at decision nodes 2 and 3, an EMV for node A is obtained as :

$$EMV(A) = 0{\cdot}70 \times (24{,}000) + 0{\cdot}30 \times (-10{,}000) = \text{Rs. } 13{,}800$$

Continuing backward, consider a new node B and C. The expected monetary values at B and C are :

$$EMV(B) = 0{\cdot}5 \times (50{,}000) + 0{\cdot}50 \times (-30{,}000) = \text{Rs. } 10{,}000, \qquad \text{and} \qquad EMV(C) = 0$$

We are now able to evaluate decision node 1. The expected net profit for this node is simply the maximum of the three banches leading into it, *i.e.*

$$EMV(1) = \text{Max. } \{13800, 10000, 0\} = \text{Rs. } 13{,}800$$

After obtaining the maximum, the branches 'market' and '*drop the product*' are dominated, and optimal decision procedure represents that the 'test market' branch should be followed, an expected net profit of Rs. 13,800.

Sensitivity Analysis. Sensitivity analysis of an optimal solution becomes very essential when the actual values of input data vary from their known values, especially when subjective probabilities are involved in the decision tree problems.

Considering above example at decision node 1, perform sensitivity analysis of probability of a favourable test marketing $P(O_1)$ and the probability of a successful immediate marketing effort $P(O_2)$.

Let $P(O_1) = p$, then $P(O_2) = 1 - p$. Thus knowing the optimal decisions at decision node 2 and 3, we have

$$EMV(S_1) = 24{,}000\,p + (-10{,}000)(1 - p) = 34{,}000\,p - 10{,}000$$

Equating EMV's of courses of action S_1 and S_2 and solving for p, we get

$$34{,}000\,p - 10{,}000 = 10{,}000 \quad \text{or} \quad p = 0{\cdot}59$$

The decision *'test market'* would be preferred for any value of p exceeding 0·59. If course of action S_1 is chosen (based on given probability value, $p = 0{\cdot}70$), then the actual value of p made course of action S_2, optimal. Therefore, the decrease in *EMV* would be

$$EMV(S_2) - EMV(S_1) = 10{,}000 - (34{,}000\,p - 10{,}000) = 20{,}000 - 34{,}000\,p$$

Again, at the decision node, let q be the probability of a successful marketing (state of nature O_3), then we would be indifferent to the alternatives *'test market'* and *'market'*, if their *EMV* 's are equal. For course of action S_2, we have

$$EMV(S_2) = 50{,}000\,q + (-30{,}000)(1 - q) = 80{,}000\,q - 30{,}000$$

The indifferent value of q can be obtained by equating *EMV* 's of courses of action S_1 and S_2, *i.e.*

$$80{,}000\,q - 30{,}000 = 13{,}800 \quad \text{or} \quad q = 0{\cdot}55$$

Course of action S_1 would be optimal for $q < 0{\cdot}55$. If course of action S_1 based on the given probability value, $q = 0{\cdot}50$, then the actual value of q make the course of action S_2, as optimal. Thus, decrease in *EMV* would be

$$EMV(S_2) - EMV(S_1) = 80{,}000\,q - 30{,}000 - 13{,}000 = 80{,}000\,q - 43{,}800.$$

Example 22. *The Oil India Corporation (OIC) is considering whether to go for an offstore oil drilling contract to be awarded in Bombay High. If OIC bid, value would be Rs. 600 million with a 65 per cent chance of gaining the contract. The OIC may set up a new drilling operation or move already existing operation; which has proved successful, to a new site. The probability of success and expected returns are as follows :*

Outcome	New Drilling Operation		Existing Operation	
	Probability	Existing Operation	Probability	Expected Revenue (Rs. million)
Success	0·75	800	0·85	700
Failure	0·25	200	0·15	350

If the corporation do not bid or lose the contract, they can use Rs. 600 million to modernize their operation. This would result in a return of either 5 per cent or 8 per cent on the sum invested with probabilities 0·45 *and* 0·55. *(Assume that all costs and revenue have been discounted to present value).*

(a) Construct a decision tree for the problem showing clearly the courses of action.

(b) By applying an appropriate decision criterion recommend whether or not the oil India Corporation should bid the contract. **[Delhi (MBA) 2001; ICWA (June) 89]**

Solution.

Decision Point	Outcome	Probability	Conditional Value	Expected Value (Rs.)
D_3 : (i) Moderate	5% return	0·45	600 × 0·05 = 30	13·5
	8% return	0·55	600 × 0·08 = 48	26·4
D_2 : (i) Undertake new drilling operation	Success	0·75	800	600
	Failure	0·25	200	50
(ii) Move existing operation	Success	0·85	700	595
	Failure	0·15	350	52·5
D_1 : (i) Do not bit but moderate	5% return	0·45	600 × 0·05 = 30	13·5
	8% return	0·55	600 × 0·08 = 48	26·4
(ii) Bid	Success	0·65	650 + 647·5	843·37
	Failure	0·35	39·9	13·9
				857·34
				− 600
				257·34

Since EMV, Rs. 257·34 at node 2 is maximum, the best decision at node D_2 is to decide for bid and if successful establish a new drilling operation.

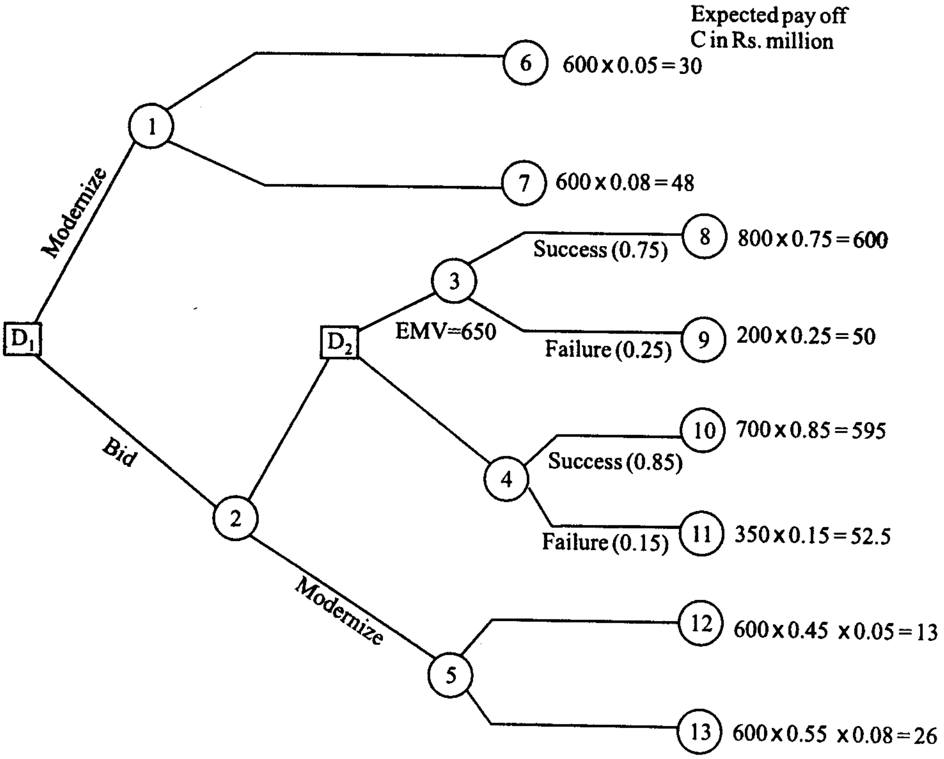


Fig. 18.5. Decision Tree.

Example 23. *Matrix company is planning to launch a new product which can be introduced initially in Western India or in the whole country. If the product is introduced only in Western India the investment outlay will be Rs. 12 million. After two years, company can evaluate the project to determine whether it should cover the whole country. For such an expansion it will have to incur an additional investment of Rs. 10 million. To introduce the product in the whole country right in the beginning would involve an outlay of Rs. 20 million. The product in any case, will have a life of 5 years, after which the plant will have zero net value. If the product is introduced only in Western India, demand would be high or low with the probabilities of* 0·8 *and* 0·2, *respectively and annual cash inflow of Rs. 4 million and Rs. 2·5 million, respectively.*

If the product is introduced in the whole country right in the beginning the demand would be high or low with the probabilities of 0·6 *and* 0·4, *respectively and annual cash inflow of Rs. 8 million and Rs. 5 million, respectively.*

Based on the observed demand in Western India, if the product is introduced in the entire country the following probabilities would exist for high and low demand on all India basis :

Western India	Whole Country	
	High Demand	Low Demand
High Demand	0·90	0·10
Low Demand	0·40	0·60

The hurdle rate applicable to this project is 12 per cent.

(a) Set up a decision tree for the investment situation.

(b) Advise Matrix company on the investment policy it should follow. Support your advice with appropriate reasoning. **[ICWA (June) 1990]**

Solution. The decision tree based on the information given in the problem is shown below in the figure.

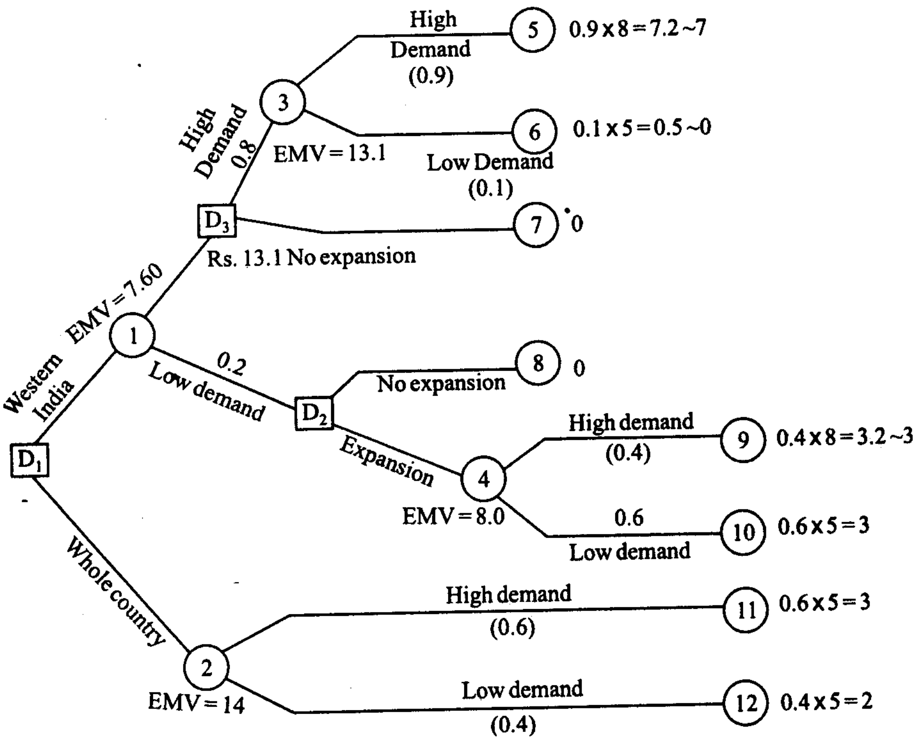


Fig. 18.6. Decision Tree.

Evaluation of Decision and Chance Nodes

Decision Points	Outcome	Probability	Conditional Value (Rs. Million)	Expected value (Rs. Million)
D_3 : (i) Expansion (ii) Stop	High Demand Low Demand	0·9 0·1	8 5	7·2 0·5 7·7 × 3 yrs = 23·1 Less cost 10·0 13·1 0 13·1
D_2 : (i) Expansion (ii) No Expansion	High Demand Low Demand	0·4 0·6	8 5	3·2 3·0 0·2 × 3 yrs = 18·6 Less cost 10·0 8·6 0 8·6
D_1 : (i) Introduction whole country (ii) Introduction in Western India (iii) Do nothing	High Demand Low Demand High Demand Low Demand	0·6 0·4 0·8 0·2	8 5	4·8 2·0 6·8 × 5 yrs = 34 Less cost 20 14 13·68 2·22 15·90 Less cost 12·00 3·90

Since EMV at node 2 is maximum, decision should be made to launch the product in the whole country.

18.9. DECISION-MAKING UNDER UTILITIES

So far, we have analysed the problems with probabilitistic states of nature where the choice 'optimal course of action was depending on the criterion of expected profit (or loss) represented in terms of monetary value. But, in many cases, such criteria related to expected monetary payoff may not be appropriate. This happens due to the fact that different individuals attach different utility to money under different situations. The term utility is the measure of preference of individuals having various alternatives available to them. The utility of a given alternative is unique to individual decision-maker and unlike a simple monetary amount which can incorporate intangible factors or subjective standards from their own value systems.

For example, *Mr. Ram has won Rs. 1,500 in a quize programme. At last, he is asked either to complete or quit. Now he has two alternatives : (a) quit and take his winnings, (b) take a last chance in which he has 50—50 chance of winning Rs. 6,000 or nothing, then the question is : what would he do. On EMV basis he has*

$$EMV(B) = 0{\cdot}50\,(6{,}000) + 0{\cdot}50\,(0) = \text{Rs. } 3{,}000$$

This amount is double the amount he has already won. But would he really give-up a sure Rs. 1,500 for a 50-50 chance of Rs. 6000 or nothing? Many individuals would not because they would think of all the things they could do with Rs. 1,500 and how they would regret it if they end-up with nothing. Hence the new payoff measure (*utility*) reflecting the decision-maker's attitude and this preference has to be introduced.

The basic axioms of utility may be noted as follows :

1. If outcome A is preferred in comparison to outcome B , then the utility $\cup\,(A)$ of outcome A , is greater than the utility, $\cup\,(B)$ of B, and vice-versa. If both are equally preferred then $\cup\,(A) = \cup\,(B)$.

2. If the decision-maker is indifferent between the two alternatives and outcome A is received with probability p and outcome C with probability $(1-p)$, then

$$\cup\,(B) = p \cup (A) + (1-p) \cup (C)$$

Under this criterion, it is assumed that a rational decision-maker will select the alternative which optimizes the *expected utility* rather than expected monetary value. Once, we know the individual's utility function alongwith the probability assigned to the outcome in a particular condition, then total expected utility for each course of action can be obtained by multiplying the utility values with their probabilities. The strategy corresponding to optimum utility function is called the *optimal strategy.*

18.9-1. Utility Function

The utility func[illegible] describes the relative preference value that individuals have for a given amount of criterion, like goods [illegible]oney, etc. They are often obtained by suggesting a condition to the subjects whereby they must select between receiving a given amount, say Rs. 25000 for certain thing versus a 50-50 chance of gaining a larger amount or nothing, say Rs. 75,000 or zero. The gamble amount Rs. 75,000 is then adjusted upward or downward until the individual is indifferent to whether decision-maker recieves the certain amount of Rs. 25000 or the gamble. Then, this indifferent point establishes it experimentally. The calculated value on the individual's utility curve and other points can be discussed similarly.

Once derived, utility function can be utilized to convert a decision criteria value into ***utils*** so that a decision can be made on the basis of maximizing the expected utility value (*EUV*) rather than, say *EMV.*

18.9-2. Utility Curve

Suppose, a loss of Rs. 1,00,000 cause a businessman to be declared bankrupt. If he was able to withstand fairly (or comfortably) the loss of Rs. 10,000, then the loss of Rs. 1,00,000 is ten times worse than the loss of Rs. 10,000. That is, the concept of utility must be developed in such a way that conversion of monetary units into utilities reflects such preferences.

Suppose the relationship between monetary gains, losses and utilities for gains and for small negative losses is obtained. As shown in Fig. 18·5, if the curve is bent down non-linearly, then we give large losses, a dis-proportionately large negative utility. Since this could lead individuals into the condition where they are attached with such a heavy weighting to the possibility of loss that they never took any risk and thus never made any gains, it is important not to make the curve bend down too steeply or to start the bending too quickly.

It is usual on the positive side of the curve to eventually bend away from the straight line. This shows that increasing units of money are resulting in smaller additional gains in utility.

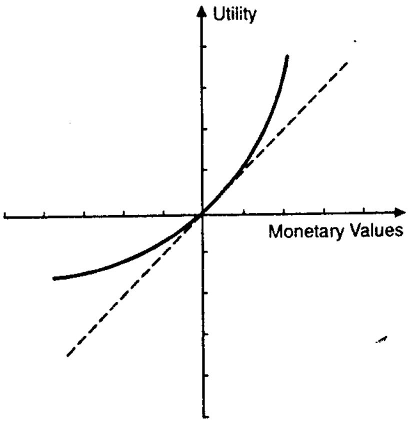


Fig. 18.7.

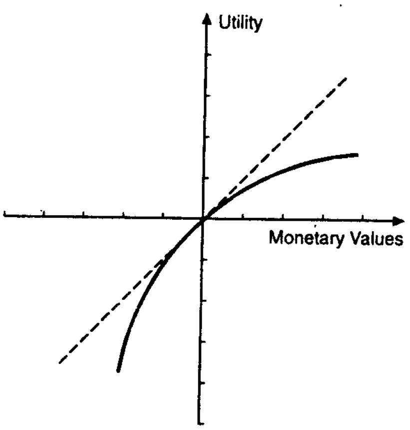


Fig. 18.8.

The general shape of utility curve is found commonly in practice as shown in Fig 18·5.

It is also possible to plot a curve bending upwards instead of downwards. Such a curve represents that the decision-maker prefers one large payoff to the financial equivalence of a number of smaller one's as given in Fig. 18·6.

18.9-3. Construction of Utility Curve

Suppose, we are given a project whose payoff is Rs. 50,000 or – Rs. 50,000. Then what probability of success would make the decision-maker just indifferent to undertake the project or not ? Suppose he says that it is required 80% probability of success to make him indifferent about under-taking the porject. Thus it is required to associate a utility value to one payoff. let the utility of Rs. 50,000 be 10, then it follows that

$$10 \times 0{\cdot}80 + \cup(-50{,}000) \times 0{\cdot}20 = 0$$

or $$8 + \cup(-50{,}000) \times 0{\cdot}20 = 0 \quad \text{or} \quad \cup(-50{,}000) = -40$$

Now we have the following three points on the utility curve;

$$\cup(+50{,}000) = +10, \quad \cup(0) = 0, \cup(-50{,}000) = -40$$

Other points on the curve can be determined by asking such questions to the decision-maker. The procedure of associating utility value to a payoff can be more realistic by attempting to relate it to either past or current project.

Example 24. *A manager must choose between two investments A and B which are calculated to yield net profits of Rs. 1,200 and Rs. 1,600, respectively, with probabilities subjectively estimated at* 0·75 *and* 0·60. *Assume the manager's utility function reveals that utilities for Rs. 1,200 and Rs. 1,600 amounts are 40 and 45 units, respectively. What is the best choice on the basis of expected utility value (EUV) ?*

Solution. The expected utility value is expressed by , $EUV = \Sigma P_i U_i$,

where P_i = probability of state of nature i, U_i = utility value of state of nature i.

Then $EUV(A) = P_A U_A = 0{\cdot}75 \times 40 = 30$ utils, $EUV(B) = P_B U_B = 0{\cdot}60 \times 45 = 27$ utils

Since $EUV(A) > EUV(B)$, best choice is the investment A .

18.10. POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS

The search and evaluation of decison alternatives often give new information. If new information is concerned with the identification of alternatives, there are revisions and expansions of the test of alternatives.. If it is concerned with the effects, the conseqences are restated. If the uncontrollable factors are involved, either the states of nature themselves are reconsidered or their likelihoods are revised. The value of new information is obtained in its impact on the expected payoff. The expected value and the cost of new information are compared to obtain whether it is worth acquiring.

An initial probability statement used to evaluate expected payoff is called a *prior* probability distribution. If it is revised in the light of information which has come to hand is called a *posterior* probability distribution. It will be obvious that the posterior probability to one sequence of state of nature becomes the prior to others which are yet to happen.

This section deals with the method after computing posterior probabilities from prior probabilities using a mathematical formula called *Baye's theorem.* A further analysis using these probabilities with respect to new expected payoffs with additional information is called *prior-posterior analysis.*

The general form of *Baye's theorem* can be drived as follows :

Let A_1 , A_2 , ... , A_n are mutually exclusive and collectively exhaustive outcomes. The prior probabilities $P(A_1)$, $P(A_2)$, ... , $P(A_n)$ are given. There exists an experimental outcome B for which the conditional probabilities $P(B \mid A_1)$, $P(B \mid A_2)$, ... $P(B \mid A_n)$ are also given. If the information that outcome B has occurred is given, then the revised probabilities $P(A_i \mid B)$, $i = 1, 2, \ldots , n$ are determined as follows :

The conditional probability is given by

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

where $$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B).$$

Since each joint probability can be represented by the product of a given marginal (prior) and conditional probability,

$$P(A_i \cap B) = P(A_i) \times P(B \mid A_i)$$

then $$P(A_i \mid B) = \frac{P(A_i)\, P(B \mid A_i)}{P(A_i)\, P(B \mid A_i) + \ldots + P(A_n)\, P(B \mid A_n)} = \frac{P(A_1)\, P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}$$

Example 25. *A company is considering the introduction of new product to its existing product range. It has defined two levels of sales as 'high' and 'low' on which to base its decision and has estimated the chances that each market level will occur, together with their costs and consequent profits or losses. This information is summarized below :*

States of nature	*Probability*	Courses of action	
		Market product (Rs. '000)	Not to market product (Rs. '000)
High sales	0·3	150	0
Low sales	0·7	– 40	0

The company's marketing manager suggests a market research survey be undertaken to provide further information on which to base the decision. On past experience with a certain market research organization, the marketing manager assesses its ability to give good information in the light of subsequent actual sales achievements as follows :

Market research survey outcome	Actual sales	
	Market 'high'	Market 'low'
'High' sales forecast	0·5	0·1
Indecisive survey report	0·3	0·4
'Low' sales forecast	0·2	0·5

Given that to undertake the market research survey will cost Rs. 20,000. State whether or not there is a case for employing the market research organization.

Solution. The expected monetary value (*EMV*) for each course of action is given in Table 18.31.

Table 18.31

States of nature	Probability	Courses of action		Expected profit (Rs. '000)	
		Market product	Not market product	Market product	Not market product
High sales	0·3	150	0	45	0
Low sales	0·7	– 40	0	– 28	0
			EMV	17	0

With no additional information, the company selects a course of action *'market product'*. But, if the company had the perfect information about *'low sales'*, then it would not go a head because the expected value is – Rs. 28,000. Thus, the value of perfect information is the expected value of low sales.

Let the states of nature be : O_1 = high sales, O_2 = indecisive report, O_3 = low sales.

Courses of action are : S_1 = high market, S_2 = low market

The computations of prior probabilities of forecast are shown in Table 18.32.

Table 18.32

States of nature O_1	Sales production	
	Market 'high' (S_1)	Market 'low' (S_2)
High sales (O_1)	$P(O_1/S_1)$ = 0·5	$P(Q_1/S_2)$ = 0·1
Indecisive report (O_2)	$P(O_2/S_1)$ = 0·3	$P(O_2/S_2)$ = 0·4
Low sales (O_3)	$P(O_3/S_1)$ = 0·2	$P(O_3/S_2)$ = 0·5

With this additional information, the company can revise the prior outcome probabilities to determine posterior probabilities. These can also be used to recalculate the *EMV* and to obtain the optimal course of action given the additional information.

Table 18.33. Calculation of revised prob. given the sales forecast

States of nature	Prior probability	$P(O_1 \cap S_1)$	$P(O_2 \cap S_1)$	$P(O_3 \cap S_1)$
High sales	$P(O_1/S_1)$ = 0·5	0·15		
	$P(O_2/S_1)$ = 0·3		0·09	
	$P(O_3/S_1)$ = 0·2			0·06
Low sales	$P(O_1/S_2)$ = 0·1	0·07		
	$P(O_2/S_2)$ = 0·4		0·28	
	$P(O_3/S_2)$ = 0·5			0·35
Marginal Probability		0·22	0·37	0·41

The posterior probabilities of actual sales given the sales forecast are :

$$P(S_1/O_1) = \frac{P(S_1)\,P(O_1/S_1)}{P(O_1)} = \frac{0{\cdot}3 \times 0{\cdot}5}{0{\cdot}22} = 0{\cdot}68.$$

Similarly, $P(S_1/O_2) = 0{\cdot}24$, $P(S_1/O_3) = 0{\cdot}14$, $P(S_2/O_1) = 0{\cdot}31$, $P(S_2/O_2) = 0{\cdot}75$. $P(S_2/O_3) = 0{\cdot}85$.

Now for each outcome, the revised probabilities are used to determine the net expected value if the additional information is supplied by the outcome shown in Table 18.34.

Table 18.34

States of nature Actual sales	*Sales forecast*						
	Revised conditional profit (Rs.)	High		Indecisive		Low	
		Prob.	*EV* (Rs.)	Prob.	*EV* (Rs.)	Prob.	*EV* (Rs.)
High	130	0·68	88·66	0·24	31·59	0·14	18·98
Low	– 60	0·31	– 19·08	0·75	– 45·42	0·85	– 51·24

Expected value of sale forecast	:	69·58	– 13·83	– 32·26
Probability of occurrence	:	0·22	0·37	0·41
Net expected value (Expected value × Prob.)	:	15·31	– 5·12	13·23

Example 26. *A farmer is attempting to decide which of three crops he should plant on his one-hundred acre farm. The profit from each crop is strongly dependent on the rainfall during the growing season. He has categorized the amount of rainfall as substantial, moderate or light. He estimates his profit for each crop as shown in the table :*

Rainfall	Estimated profit (Rs.)		
	Crop A	Crop B	Crop C
Substantial	7,000	2,500	4,000
Moderate	3,500	3,500	4,000
Light	1,000	4,000	3,000

Depending on the weather in previous seasons and the current projection for the coming season, he estimates the probability of substantial rainfall as 0·2 , *that of moderate rainfall as* 0·3 *and that of light rainfall as* 0·5.

Furthermore, services of forecasters could be employed to provide a detailed survey of current rainfall prospects as given in the table below :

Rainfall Prediction

Actual rainfall	Substantial	Moderate	Light
Substantial	0·70	0·25	0·05
Moderate	0·30	0·60	0·10
Light	0·10	0·20	0·70

(a) From the available data, determine the optimal decision as to which crop to plant.

(b) Determine whether it would be economical for the farmer to hire the services of a forecaster.

Solution. *(a)* Let O_j be the states of nature ($j = 1, 2, 3$) representing '*substantial rainfall*', '*moderate rainfall*' and '*light rainfall,*' respectively and S_i be the courses of action ($i = 1, 2, 3$) representing '*Crop A*', '*Crop B*' and '*Crop C*', respectively.

The calculation of *EMV*'s is given in Table 18.35.

Table 18.35.

States of nature	Prior probability	Conditional profit (Rs.)			Expected profit (Rs.)		
		Courses of action			Courses of action		
O_j	$P(O_j)$	S_1	S_2	S_3	S_1	S_2	S_3
O_1	0·2	7,000	2,500	4,000	1,400	500	800
O_2	0·3	3,500	3,500	4,000	1,050	1,050	1,200
O_3	0·5	1,000	4,000	3,000	500	2,000	1,500
				EMV	2,950	3,550	3,550

The maximum *EMV* is Rs. 3,550, therefore optimal course of action is S_2 *i.e.* plant *crop B*. Observe that it would make no sense to plant more than one kind of crop, since the maximum expected profit is clearly obtained by planting all 100 acres with crop *B*. As given, in case of *crop B*, the outcome O_1 (substantial rainfall) occurred 20% of the time, outcome O_2 (moderate rainfall) occurred 30% of the time and outcome O_3 (light rainfall) occurred 50% of time, then we would expect an average profit of Rs. 3,550. The other two *EMV*'s can be interpreted in a similar fashion.

(b) Let B_1 = forecast of substantial rainfall, B_2 = forecast of moderate rainfall,

B_3 = forecast of light rainfall.

The likelihood values are given in Table 18.36.

Table 18.36

States of nature O_j	Forecast likelihood		
	$P(B_1/O_j)$	$P(B_2/O_j)$	$P(B_3/O_j)$
O_1	0·70	0·25	0·05
O_2	0·30	0·60	0·10
O_3	0·10	0·20	0·70

From Table 18.36, where the maximum profit for each state of nature is *underlined*, the expected profit with perfect information is.

$$EPPI = 0{\cdot}2(7{,}000) + 0{\cdot}3(4{,}000) + 0{\cdot}5(4{,}000) = \text{Rs. } 4{,}600.$$

We then have

$$EVPI = EPPI - EMV = (4{,}600 - 3{,}550) = \text{Rs. } 1{,}050.$$

For each of the three possible states of nature (forecast results), revise the prior probabilities to determine the revised EM values as given in table 18.37.

Table 18.37

State of nature O_j	Prior prob. $P(O_j)$	States of nature B_i	Conditional prob. $P(B_i/O_j)$	Joint probability $P(B_i \cap O_j) = P(O_j)\,P(B_i/O_j)$		
O_1	0·2	B_1	0·70	0·14	—	—
		B_2	0·25	—	0·05	—
		B_3	0·05	—	—	0·01
O_2	0·3	B_1	0·30	0·09	—	—
		B_2	0·60	—	0·18	—
		B_3	0·10	—	—	0·03
O_3	0·5	B_1	0·10	0·05	—	—
		B_2	0·20	—	0·10	—
		B_3	0·70	—	—	0·35
Marginal probability				0·28	0·33	0·39

The required posterior probabilities are determined as given in Table 18.38.

Table 18.38

State of nature (B_i)	Probability $P(B_i)$	States of nature O_j	Posterior probability $P(O_j/B_i) = P(B_i \cap O_j)/P(B_i)$
B_1	0·28	O_1	0·14/0·28 = 0·50
		O_2	0·09/0·28 = 0·32
		O_3	0·05/0·28 = 0·18
B_2	0·33	O_1	0·05/0·33 = 0·15
		O_2	0·18/0·33 = 0·55
		O_3	0·10/0·33 = 0·30
B_3	0·39	O_1	0·01/0·39 = 0·02
		O_2	0·03/0·39 = 0·02
		O_3	0·35/0·39 = 0·90

Now for each state of nature, the revised probabilities are used to recalculate the *EMV* 's given the additioanl information supplied by that state of nature as shown in Table 18.39.

Table 18.39

State of nature	Forecasted outcome								
	B_1			B_2			B_3		
	Prob.	*COL*	*EOL*	Prob.	*COL*	*EOL*	Prob.	*COL*	*EOL*
O_1	0·50	0	0	0·15	500	75	0·02	3000	60
O_2	0·32	4500	1440	0·55	500	275	0·08	0	0
O_3	0·18	3000	540	0·30	0	0	0·90	1000	900
Posterior EOL			: 1980			350			960

The expected value of sample information can be determined by multiplying posterior *EOL*'s with revised probabilities as given below :

State of nature (B_i)	Probability $P(B_i)$	Expected opportunity loss (*EOL*)	*EVSI*
B_1	0·28	1980	554·4
B_2	0·33	350	115·5
B_3	0·39	960	374·4
			1044·3

The expected value of sample information, *i.e.* Rs. 1044·3 indicates the money which the farmer has to pay for hiring the services of a forecaster.

Q. 1. Given the complete set of outcomes in a certain situation, how is the EMV determined for a specific course of action ? Explain in your own words.

2. What is scientific decision making process ? Discuss the role of statistical method for it.

3. Explain the difference between expected opportunity loss and expected value of perfect information.

4. Describe some methods which are useful for decision-making under uncertainty. Illustrate each by an example.
5. (a) Indicate the difference between decision-making under risk, and uncertainty in statinsfical decision theory.
   (b) Explain the various quantitative methods which are useful for decision-making under uncertainty.
6. Describe a business situation where a decision-maker faces a decision under uncertainty and where a decision based on maximizing the expected monetary value cannot be made. How do you think the decision-maker would make the required decision ?
7. In a situation involving a probabilistic distribution of outcomes, where the monetary value of each course of action is a known value for each of several possible decisions, why is it not always reasonable to choose that decision which maximize EMV ?
8. Briefly explain the different decision rules usually adopted in context of decision-making under condition of uncertainty. **[Poona (M.B.A.) 92]**

EXAMINATION PROBLEMS

1. A *toy company* is bringing out a new type of toy. The company is attempting to decide whether to bring out a full, partial, or minimal product line. The company has three levels of product acceptance and has estimated their probability of occurrence.. Managemnet will make its decision on the basis of maximizng the expected profit from the first year of production. The relevent data are shown in the following table :

First-year Profit (Rs. '000)

Product acceptance	Probability	Product line		
		Full	Partial	Minimal
Good	0·2	80	70	50
Fair	0·4	50	45	40
Poor	0·4	− 25	− 10	0

(a) What is the optimal product line and its expected profit ?

(b) Develop an opportunity loss table and calculate the EOL values. What is optimal value of EOL and the optimal course of action ?

[**Ans.** (a) Partial : Rs. 28,000, (b) Rs. 8,000]

2. Calculate the loss table from the following payoff table :

Action	Events			
	E_1	E_2	E_3	E_4
A_1	50	300	− 150	50
A_2	400	0	100	0
A_3	− 50	200	0	100
A_4	0	300	300	0

Suppose that the probabilities of the events in this table are :

$P(E_1) = 0{\cdot}15$, $P(E_2) = 0{\cdot}45$, $P(E_3) = 0{\cdot}25$, $P(E_4) = 0{\cdot}15$.

Calculate the expected payoff and expected loss to each action.

[**Ans.** Expected payoff : $E(A_1) = 112{\cdot}5$; $E(A_2) = 85{\cdot}0$, $E(A_3) = 97{\cdot}5$, $E(A_4) = 210{\cdot}0$.

Expected loss : $A_1 \rightarrow 172{\cdot}5$, $A_2 \rightarrow 200$; $A_3 \rightarrow 187{\cdot}5$; $A_4 \rightarrow 75$]

3. A milkman buys milk at Rs. 2 per litre and sells for Rs. 2·50 per litre. Unsold milk has to be thrown away. The daily demand in litres has the following probabilistic distribution :

Litres :	46	48	50	52	54	56	58	60	62	64
Probability :	0·01	0·03	0·06	0·1	0·2	0·25	0·15	0·1	0·05	0·05

If each day's demand is independent of previous day'show many litres should be ordered every day ?

[**Ans.** 52 or 54]

4. A certain output is manufactured at Rs. 2 and sold at Rs. 4 per unit. The product is such that if it is produced but not sold during a week's time, it becomes worthless. The weekly sales record in the past are as follows :

Demand per week :	20	25	40	60
Number of weeks :	5	15	25	5

Suggest the optimal act which should be taken by the manufacturer of the output.

5. A news agency receives its weekly order of a magazine on monday and cannot reorder. Each copy costs Rs. 1·80 and sells for Rs. 3·00. Unsold copies may be reutrned the following week for a Rs. 1·20 rebate. When the agency runs out of copies and cannot supply a customer, it estimates its 'good will' loss as Rs. 2·40 in future profits, figuring that the

customer will take his business elsewhere for a couple of weeks, on the average. Demand has been remarkably constant, ranging between 28 and 40 copies, as shown below :

Demand, copies	:	28	32	36	40
Fraction of time	:	0·30	0·40	0·20	0·10

(a) Construct a payoff table and use it to determine the optimal number of copies to stock and the expected profit.

(b) Construct an opportunity loss table and find the optimal number of copies to stock and optimal value of EOL.

(c) How much would it be worth to know the exact demand each week ? What would be the expected profit if this were possible ?

[**Ans.** (a) 36; Rs. 8·76, (c) Rs 0·96; Rs. 9·72]

6. A car manufacturer uses a special control device in each car he produces. Two alternative methods can be used to detect and avoid a faulty device. Under the first method, each device is tested before it is installed, and all faulty devices are deiscovered before installation. The cost is Rs. 2 per test. Alternatively, the control device can be installed without being tested, and a faulty device can be detected and removed after the car has been assembled, at a cost of Rs. 20 per faulty device.

Regardless of which method is used, faulty devices cannot be repaired and must be discarded.

A manufacturer purchases the control devices in batches of 10,000. Based on past experience, he estimates the proportion of defective components and the associated probability to be :

Proportion of faulty devices	:	0·80	0·12	0·16
Probability	:	0·20	0·70	0·10

(a) Which inspection method should the manufacturer adopt ?

(b) What is the expected value of perfect information (EVPI) ?

[**Ans.** (a) First alternative; Rs. 20,000

(b) $EVPI$ = (Expected cost under uncertainty) − (Expected cost with perfect information)

= 20,000 − (16,000 × 0·2 + 20,000 × 0·7 + 20,000 × 0·1) = 20,000 − 19,200

= Rs. 800 per batch]

7. A well known departmental store advertises female fashion garments from time to time in the Sunday press. Only one garment is advertised on each occasion. Experience of garments in the price range Rs. 30–40 leads the management to assess the statistical probability of demand at various levels, after each advertisement as follows :

Demand no. of garments	One newspaper	After advertising in two newspapers
30	0·10	0·00
40	0·25	0·15
50	0·40	0·35
60	0·25	0·40
70	0·00	0·10

The next garment to be advertised to sell at Rs. 35 will cost Rs. 15 to make and can be disposed of to the trade, if unsold, for Rs. 10. The store's advertising agents will charge Rs. 37·50 for art work and block making for advertisement, and each newspaper will charge Rs. 50 for the insertion of the advertisement. You are required to :

(a) Calculate how many garments to the nearest 10 should be purchased by the stores in order to maximize expected gross profit, after advertising in : (i) one newspaper (ii) two newspapers.

(b) Calculate the amount, if any, of the expected net profit after advertising costs (i) and also under a (ii).

[**Ans.** Marginal profit (MP) = Rs. (35 − 15) = Rs. 20 , Marginal loss (ML) = Rs. (15 − 10) = Rs. 5

Conditional payoff = MP × garment sold − ML × garment unsold

(a) (i)$EMV(S_1)$ = Rs. 600 ; $EMV(S_2)$ = Rs. 775; $EMV(S_3)$ = Rs. 887·5 ; $EMV(S_4)$ = Rs. 900 ; $EMV(S_4)$ = Rs. 850

Max. EMV is corresponding to course of action S_4 *i.e.* purchase 60 garments.

(ii)$EMV(S_2)$ = Rs. 800 ; $EMV(S_3)$ = Rs. 962·5 ; $EMV(S_4)$ = Rs. 1,037·5 ; $EMV(S_5)$ = Rs. 1,012·5.

Max. EMV is corresponding to course of action S_4 *i.e.* purchase 60 garments.]

8. A manufacturer's representative has been offered a new product line. If he accepts the new line, he can handle it in one of the two ways. The best way according to the manufacturer would be to set a separate sales-force to handle the new line exclusively. This would involve an initial investmnet of Rs. 1,00,000 in the office, equipment and the hiring and training of the salesman. On the other hand, if the new line could be handled by the existing sales force using the existing facilities, the initial investment would only be Rs. 30,000, principally for training of his present salesman.

The new product sells for Rs. 250. The representative normally receives 20% of the sale price on each unit sold of which 10% is paid as commission to the salesman. In order to encourage the manufacturer's representative to set-up a separate sales organization to handle the new product, manufacturer offers to pay 30% of the sale price of each unit sold to the representative if the representative sets up a separate sales organization. Otherwise, the normal 20% will be paid. In either case the saleman gets a 10% commission.

Based on the size of the territory and experience with other products, the representative estimates the following probabilities for annual sales of the new prdouct :

Sales in Units :	1,000	2,000	3,000	4,000	5,000
Probability :	0·10	0·15	0·40	0·30	0·05

(a) Set-up a regret table.

(b) Find the expected regret of each course of action.

(c) Which course of action would have been best under the maximum criterion ?

[**Hint :** Let S_1 = course of action for representative to install new states facilities

S_2 = course of action to continue with existing sales facilities. Therefore, payoff function corresponding to S_1 and S_2 would be :

$$S_1 = -\ 1{,}00{,}000 + 250 \times \frac{30 - 10}{100} \times \alpha = -\ 1{,}00{,}000 + 50\,\alpha$$

$$S_2 = -30{,}000 + 250 \times \frac{20 - 10}{100} \times \alpha = -30{,}000 + 25\,\alpha$$

Equating the two, we get $-1{,}00{,}000 + 50\,\alpha = -30{,}000 + 25\,\alpha$, $\alpha = 2{,}800$]

9. A grocer is faced with a problem of how many items to be stocked to meet tomorrow's demand

Purchase price = Rs. 8 per item, Selling price = Rs. 10 per item

Total demand of items per day	:	25	26	27	28
Number of days each demand levels was recorded	:	20	60	100	20

(a) What should be the optimal decision of the grocer concerning the items to be stocked ? Assuming that the grocer has a perfect knowledge, what should be his expected profits.

(b) Briefly explain the different decision rules usually adopted in context of decision-making under conditions of uncertainty ?

10. A mineral processing company wants to decide about the number of spare gear trains it has to order out at the time of placing order for a high horse power gear box connected to a grinding unit. Although the life of a gear can be as high as 30 years and more, sudden failures cannot be ruled out. In case of failure it would be expensive and time consumming to get a spare gear train. The cost would be Rs. 2,00,000 including the loss of production due to down time of the equipment. If ordered-out with gear box, a gear train would cost only Rs. 10,000 per unit. The following data is based on an analysis of past experience of 100 gear boxes.

Number of spare gear trains required	:	0	1	2	3	4
Number of gear boxes requiring the spares	:	93	4	2	1	0

You are expected to advise the optimal order size.

11. *XYZ* company dealing with newly invented telephonic device is faced with the problem of selecting the following courses of action available :

(i) Manufacture the device himself; or (ii) be paid on a royalty basis by anotehr manufacturer; or

(iii) sell the rights for his invention for a lump sum ?

The profit (in Rs. 1,000's) which can be expected in each case and the probabilities associated with the level of sale are shown in the following table :

Outcome	Probability	Manufacture himself	Royalties	Sell all rights
High sales	0·1	75	35	15
Medium sales	0·3	25	20	15
Low sales	0·6	– 10	10	15

Represent the company's problem in the form of a decision tree. Redraw further the decision tree by introducing the following additional inforamtion :

(a) If manufactures himself and sales are medium or high, then company has the opportunity of developing a new version of its telephone.

(b) From the past experience, company estimates that there is a 50% chance of successful development,

(c) The cost of development is Rs. 15 and th returns after deducting the development cost are Rs. 30 and Rs. 10 for high and medium sales respectivley.

[**Ans.** Royalties; 3 MV = Rs. 15·5 If p is the probability of high sell, then (i) manufacture himself when $p > 0{\cdot}24$ (ii) paid on royalty basis when $0{\cdot}07 < p < 0{\cdot}24$ (iii) sell all rights for lump sum when $p < 0{\cdot}07$]

12. The Hi-Bounce Company manufacture guaranteed tennis balls. At the present time, apprximately 10% of the tennis balls are defective. A defective ball leaving the factory costs the company Rs. 0·50 to honour its guarantee. Assume that all defective balls are returned. At a cost of Rs. 0·10 per ball, the company can conduct a test, which always correctly identifies both good and bad tennis ball.

(a) Draw a decision tree and determine the optimal course or action and its expected cost.

(b) At what test cost should the company be indifferent to resting :

[**Ans.** (a) Do not test; Rs. 0·05 ; (b) 0·50]

13. A company manufacturing large electrical equipment is anticipating the possibility of a total or a partial copper strike in the near future. It is attempting to decide whether to stockpile a large amount of copper, at an additioanl cost of Rs. 50,000; a small amount, costing an additional Rs. 20,000; or to stockpile no additional copper at all. The stockpiling costs, consisting of excess storage, holding and handling costs and so forth, are over and above the actual material costs.

If there is a partial strike, the company estimates that an additional cost of Rs. 50,000 for delayed orders will be incurred only if there is no stockpile at all. If a total strike occurs, the cost of delayed orders is estimated at Rs. 1,00,000 if there is only a small stockpile and Rs. 2,00,000 with no stockpile. The company estimates the probability of a total strike as 0·1 and that of a partial strike as 0.
(a) Develop a coditional cost table showing the cost of all outcome and course of action combinations.
(b) Determine the preferred course of action and its cost. What is EPPI ?
(c) Develop a coditional opportunity loss table.
(d) Without calculating EOL, find EVPI.
[**Ans.** (b) Small stockpile ; Rs. 30000, – Rs. 11000, (d) Rs. 19000]

14. An automobile owner faces the decision as to which deductible amount of comprehensive insurance coverage to selct. Comprehensive coverage includes loses due to fire, vehicle theft, vanadalism and the forces of nature. The possible choices are zero deductible coverage for Rs. 60 per year or Rs. 50 deductible ocverage for Rs. 45 per year. (The owner pays the first Rs. 50 of any loss of atleast Rs. 50.) Considering incidents covered by the comprehensive portion of the policy, the owner feels that the annual chances of more than one such incident the owner believes that it will be at atleast Rs. 50.

Some of the owner's utility values are given below :

Amount (Rs.) :	– 95	– 60	– 50	– 45	0
Utility :	0·2	0·4	0·45	0·47	0·5

(a) Sketch the owner's utility curve. Is the owner a risk avoider, an EMV'er or a risk taker ?

(b) On the basis of the utility curve drawn in part (a) , should the owner take the zero deductible or the Rs. 50 deductible comprehensive coverage ?

15. Consider the following table which present the monetary payoff associated with three farming plans given three states of nature.

Farming plan (Acreage)	States of nature		
	Bad weather (O_1) (Rs.)	Average weather (O_2) (Rs.)	Ideal weather (O_3) (Rs.)
Plant small	– 10,000	15,000	25,000
Plant half	– 50,000	25,000	50,000
Plant entire	–1,00,000	40,000	90,000

Conversation with the farmer has led to the following assessment of his monetary payoffs and associated utility indices :

Point	Monetary payoff	Utility index
1	– 1,00,000	0·0
2	90,000	1·00
3	55,000	0·50
4	75,000	0·75
5	25,000	0·25

Use these utility indices to construct a utility function. Also determine a utility table to replace the payoff table shown above.

Assume that the farmer has estimated that there is a 0·25, 0·50 and 0·25 probability associated with 'bad', 'average' and 'ideal' weather respectively. Determine the estimated monetary values and estimated utilities for the three farming plans. What is the best decision based on an expected monetary value decision criterion ? What is the best decision based on an expected utility decision criterion ?

16. In the *toy manufacturing company* (Problem 1), suppose that the product acceptance probabilities are not known, so that the available information is as shown below :

Product acceptance	Anticiptated first-year profit (Rs. 1,000's)		
	Product line		
	Full	Partial	Minimal
Good	80	70	50
Fair	50	45	40
Poor	– 25	– 10	0

Determine the optimal decision under each of the following decision criteria and show how you arrived at it :
(a) Maximax, (b) Maximin, (c) Equal likelihood, (d) Minimax regret.
[**Ans.** (a) Full, (b) Minimal, (c) Full or Partial, (d) Partial]

17. In above Problem if the company has no real feeling for the prob. of either a total or partial strike, then what decision would they make under each of the following decision criteria ? State how your decision was determined.
(a) Maximax, (b) Maximin, (c) Equal likelihood, (d) Minimax regret.
What do you feel it is the 'best' decision in this situation ? Explain
[**Ans.** (a) No stockpile, (b,c,d) large stockpile]

18. Two companies, *Hundustan Electro Carbon ltd.* and *Poly-Chemicals Ltd.* expect to announce plans for next year's operations on the same day. One vital issue that the share-holders of each company, as well as the general public have an interest in, the position that each of the companies will take regarding the problem of pollution. If one company, for example, declared its intent to take action towards stopping pollution, its public image will be greatly improved. But on the other hand, such action could increase its costs and put it in a bad position with respect to its competitor, if the competitor chooses not to take the same course of action. Each company can take any of the following three actions :
(a) adoption of a policy towards ending pollution, *(b)* complete avoidance of the issue, or
(c) intention to continue as in the past.
The payoffs for the action are :

	Poly-chemicals ltd.		
	Action *(a)*	Action *(b)*	Action *(c)*
Hindustant action (a)	3	-2	4
Electro action (b)	-1	4	2
Carbon ltd. action (c)	2	2	6

Determine the optimal course of action for each company.

19. Suppose a company has several independent investment opportunities each of which has an equal chance of gaining Rs. 1,00,000 or lossing Rs. 60,000. What is the probability that the company will lose money on two such investments ? On three such investmnets ? On four such investments ?

If a company has a number of independent investment opportunities, in each of which the financial risk is relatively small, compared to its overall asset position, why should the company try to maximize EMV, rather than expected utility ?
[**Ans.** 0·25; 0·50; 0·3125]

20. The *Ore Mining Company* is attempting to decide whether or not a certain piece of land should be purchased. The land cost is Rs. 3,00,000. If there are commercial ore deposits on the land, the estiamted value of property is Rs. 5,00,000. Iff no ore deposits exist, however, the property value is estimated at Rs. 2,00,000. Before purchasing the land, the property can be cored at a cost of Rs. 20,000. The coring will indicate if conditions are favourable or unfavourable for ore mining. If the coring report is favouable, the probability of recoverable ore deposits on the land is 0·8, while if the coring report is unfavourable the probability is only 0·2. Prior to obtaining any coring information, management estimates that the odds are 50—50 that ore is present on the land. management has also received coring reports on places of land similar to the one in question and found that 60% of the coring reports were favourable.

Construct a decision tree and determine whether the company should purchase the land, decline to purchase it, or take a coring test before making its decision. Specify the optimal course of action and EMV.
[**Ans.** Test ; If favourable—purchase; If not—do not purchase, Rs. 64,000]

21. A company which operates a chain of lunch plans to install a unit in either of two locations. The company feels that the probability of a unit being successful in location *X* is ¾ and that, it is successful it will make a annual profit of Rs. 4,00,000, if it is not successful the company will lose Rs. 1,00,000 per year. The prob. of a unit making success in location *Y* is only ½ but if it does succeed, the annual profit will be 6,00,000. If at does not succeed in location *Y*, the annual loss will be Rs. 1,20,000. Where should the company locate the new unit so as to maximize its expected gain ?

22. A businessman has two independent investments *A* and *B* available to him, but he lacks the capital to undertake both of them simultaneously. He can choose to take *A* first and than stop, or if *A* is successful them take *B*, or vice-versa. The prob. of success of *A* is ·07, while for *B* is is 0·4. Both investments require an initial capital outlayof Rs. 2,000 and both return nothing of the venture is unsuccessful. Successful completion of *A* will return Rs. 3,000 (over cost) and successful completion of *B* will return Rs. 15,000 (over cost). Draw the decision tree and determine the best strategy.
[**Ans.** The best strategy is to accept *A* first, and of it is successful then accept *B*]

23. **(a)** A small industry finds from the past data, that the cost of making an item is Rs. 25, the selling price of an items is Rs. 30, if it is sold within a week, and it could be disposed at Rs. 20 per item at the end of the week :

Weekly sales :	< 3	4	5	6	7	≥ 8
No. of weeks :	0	10	20	40	30	0

Find the optimum number of items per week the industry should produce.

[**Hint.** Cost of producing an item = Rs. 25, selling price = Rs. 30, profit on sale = Rs. (30 – 25) = Rs. 5 ; if unsold, disposable price = Rs. 20. From this information, the various conditional profit (payoff) values for each act-event combination are given as follows :

$$\text{Conditional profit value} = \text{MP} \times \text{item sold} - \text{ML} \times \text{item unsold}$$
$$= (30 - 25) \text{ items sold} - (25 - 20) \text{ items unsold}$$
$$= \begin{cases} (30-25)\, S = 5\, S\,;\ \text{if } P \leq S \\ (30-25)\, S - (25-25)\,(P-S) + 10\, S - 5\, P,\ \text{if } P > S \end{cases}$$

where S = number of items sold
P = number of items produced.

(b) Under an employment promotion scheme, it is proposed to allow sale of newspaper on the buses during off peak hours. A vendor can purchase the newspaper at a concessional rate of 75 paise per copy and can sell it for one rupee. Copies unsold at the end of the day are, however, a dead loss. The demand probability distribution has been estimated as follows :

Demand :	160	170	180	190	200	210
Probability :	0·04	0·19	0·33	0·26	0·11	0·07

How many copies should the vendor order so as to maximize his expected profit ? **[Delhi (M. Com.) 99]**

24. **(a)** A certain output is manufactured at Rs. 8 and sold at Rs. 14 per unit. The product is such that if it is produced but not sold during a day's time it becomes in the past are as follows :

Demand per day :	30	40	50	60	70
No. of days each sales level was recorded :	24	24	36	24	12

(i) Prepare a pay-off and a regret table (ii) Find the expected pay-offs and regret

(iii) Find the optimum act and EVPI.

[Hint. (a) Conditional value = MP × (item sold) – ML × (item unsold) = Rs. (14 – 8) – Rs. 8

Pay-off and Regret Table

Event	Prob.	Conditional pay-off (Rs.) Act					Conditional opportunity loss (Rs.)				
		$A_1 : 30$	$A_2 : 40$	$A_3 : 50$	$A_4 : 60$	$A_5 : 70$	$A_1 : 30$	$A_2 : 40$	$A_3 : 50$	$A_4 : 60$	$A_5 : 70$
$E_1 : 30$	0·20	180	100	20	– 60	– 140	0	80	160	240	320
$E_2 : 40$	0·20	180	240	160	80	0	60	0	80	160	240
$E_3 : 50$	0·30	180	240	300	220	140	120	60	0	80	160
$E_4 : 60$	0·20	180	240	300	360	280	180	120	60	0	80
$E_5 : 70$	0·10	180	240	300	360	420	240	180	120	60	0
		180	212	$\boxed{216}$	178	112	108	76	$\boxed{72}$	110	176
		Expected pay-off					Expected opportunity loss				

The expected value of perfect information is 72 (being the regret associated with the optimum policy] :

(b) Mr. Sumant buys a perishable commodity at Rs. 5 each. The profit per unit is Rs. 5. This perishable commodity he can keep in his shop for a week and at the end of each week the leftover are sold to a restaurant for Rs. 3 each (at a loss of Rs. 2 each). Mr. Sumant has given the record for past 100 weeks for his weekly sales are given below :

Weekly demand :	1	2	3	4	5	6	7
Number of weeks :	5	10	25	30	20	5	5

(i) Construct the conditional profit table, (ii) Determine the optimum number of units of this commodity to order weekly in order to maximize his profit (iii) Compute EPPI and EVPI (iv) Construct condition loss table, (v) Compute EOL. (vi) Compare (iii) and (v). **[Delhi (M.B.A.) March 99]**

25. The demand for a seasonal product is as given below :

Demand during the season :	40	45	50	55	60	65
Probability :	0·10	0·20	0·30	0·25	0·10	0·05

The product costs Rs. 60 per unit and sells at Rs. 80 per unit. If the units are not sold within the season, they will have no market value.

(i) Prepare a pay-off and a regret tabel, (ii) Find the expected pay-offs and regret, (iii) Find the optimum act and EVPI. **[Allahabad (M.B.A.) 95]**

[Hint.

Event (Demand)	Prob.	Act (Produce)						Act (Produce)					
		40	45	50	55	60	65	40	45	50	55	60	65
40	0·10	800	500	200	– 100	– 400	– 700	0	300	600	910	1200	1500
45	0·20	800	900	600	300	0	– 300	100	0	300	600	0	1200
50	0·30	800	900	1000	·700	400	100	200	100	0	300	600	900
55	0·25	800	900	1000	1100	800	500	300	200	100	0	300	600
60	0·10	800	900	1000	1100	1200	900	400	300	200	100	0	300
65	0·05	800	900	1000	1100	1200	1300	500	400	300	200	100	0
		EMV :						EOL :					
		800	860*	840	700	460	180	220	160*	180	221	380	840

Since the expected value is highest with 45 units (*i.e.*, 860) the optimum number of units to be produced is 45.

(ii) Expected profit with perfect information

= 800 × 0·1 + 900 × 0·2 + 1000 × 0·3 + 1100 × 0·25 + 1200 × 0·1 + 1300 × 0·05

= Rs. 1020.

∴ EVPI = Rs. 1020 – Rs. 860 = 160.

EVPI is an important concept in decision analysis. EVPI gives an upper bound of the amount which the decision maker can spend for obtaining the perfect information as to which event would occur.

26. A milk producing co-operative union desires to determine how many milegrams of butter it should produce on daily basis to meet the demand. Past records have shown the following pattern of demand :

Quantity demanded (Number of kg.) :	15	20	25	30	35	40	45
Number of days on which given level of demand occurred :	4	16	20	80	40	30	10

Assume that the stock levels are restricted to the range 15–45 kg. (a multiple of 5) and that butter left unsold at the end of the day must be disposed of due to inadequate storing facilities. Butter costs Rs. 14·00 per kg. and is sold at Rs. 20·00 per kg. (a) Construct a conditional profit table (b) Determine the action alternative associated with the maximization of expected profit (c) Determine EVPI. **[Gujarat (M.B.A.) 96]**

[Hint.

Event (Demand)	*Prob.*	*Conditional payoff (Rs.) Act (Stock)*							*Maximum conditional payoff*	*Expected payoff*
		15	*20*	*25*	$\boxed{30}$	*35*	*40*	*45*		
15	0·02	90	20	– 50	– 120	– 190	– 260	– 330	90	1·8
20	0·08	90	120	50	– 20	– 90	– 160	– 230	120	9·6
25	0·10	90	120	150	80	10	– 60	– 130	150	15·0
30	0·40	90	120	150	180	110	40	– 30	180	92·0
35	0·20	90	120	150	180	210	140	70	210	42·0
40	0·15	90	120	150	180	210	240	170	240	36·0
45	0·05	90	120	150	180	210	240	270	270	13·5
EMV		90	118	138	**148***	118	68	3		119·90

EVPI = Payoff under certainty = EMV* = Rs. (189·90 – 148) = Rs. 41·90

27. M/s. Ram Lal & Sons are faced with the problem of determining the optimum number of certain magazine to order for sale. The magazine which costs Re. 0·50 per copy sells for Re. 1·00. If the company orders more copies than it can sell, the unsold copies can be returned under the prior wholesale contract for a refund under the following formula : up to first 500 copies, refund is Re. 0·30 for each unsold copy : 501 to 1000 copies, refund is Re. 0·20 for each copy and for over 1,000 copies refund is Re. 0·10 each. The sales record of past 100 weeks is given in the following table :

No. of copies sold (per week) :	4,000	5,000	6,000	7,000	8,000
No. of weeks :	10	25	35	20	10

(i) What is the optimum decision ?

(ii) Compute the expected maximum profit. **[Nagpur (M.B.A.) 97]**

[Hint.

Computation of Expected Profit of Each Act

Event (No. of copies sold)	*Prob.*	*Act (No. of copoes ordered)*				
		A_1 : 4,000	A_2 : 5,000	A_3 : 6,000	A_4 : 7,000	A_5 : 8,000
E_1 : 4,000	0·10	2,000	1,750	1,350	950	550
E_2 : 5,000	0·25	2,000	2,500	2,250	1,850	1,450
E_3 : 6,000	0·35	2,000	2,500	3,000	2,750	2,350
E_4 : 7,000	0·20	2,000	2,500	3,000	3,500	3,250
E_5 : 8,000	0·10	2,000	2,500	3,000	3,500	4,000
Expected Profit		2,000	2,425	2617·50*	2,570	2,290

The largest expected profit accures from order of 6,000 copies.]

28. A toy camera manufacturer produces two models (*standard* and *delux*). In preparation for the heavy Christmas selling season, he must decide how many of each model to produce. Variable cost of the standard camera is Rs. 10 and selling price is Rs. 20, variable cost of the delux model is Rs. 20 and the selling price is Rs. 35. He estimates demand as follows :

Standard Model		Delux Model	
Demand	Probability	Demand	Probability
6,000	0·30	2,000	0·20
8,000	0·70	4,000	0·80

Any cameras not sold during the season are sold at salvage price of Rs. 5 for the standard and Rs. 10 for the deluxe model. The manufactuRer feels that different segments of the market purchase the two different models, thus the probabilities of sales given above are independent. Supposing unlimited production capacity, the two decisions can be made independently.

What are the optimum quantities of each model to produce ? What are the two optimum EMV's. **[M.D. (M.B.A.) 95]**

[**Hint.** **Standard Model** **Delux Model**

Event (Demand)	*Prob.*	*Expected Profit (Rs.) Act (Produce)*		*Event (Demand)*	*Prob.*	*Expected Profit (Rs.) Act (Produce)*	
		6,000	8,000			2,000	4,000
6,000	0·3	60,000	50,000	2,000	0·2	30,000	10,000
8,000	0·7	60,000	30,000	4,000	0·8	30,000	60,000
EMV		60,000	**71,000***			30,000	**50,000***

29. An engineering firm has installed a machine costing Rs. 4 lacs and is in the process of deciding on an appropriate number of a certain spare parts required for repairs. The spare parts cost Rs. 4000 each but are available only if they are ordered now. In case the machine fails and no spares are available, the cost to the company of mending the plant would be Rs. 18,000. The plant has an estimated life experience with similar machines, is as follows :

No. of failures during 8 yearly period :	0	1	2	3	4	5	6+
Probability :	0·1	0·2	0·3	0·2	0·1	0·1	0

Ignoring any discounting for time value of money, determine the following :

(a) The optimal number of units of the spare part on the basis (i) minimax principle, (ii) minimum principle, (iii) Laplace principle, and (iv) expected cost principle.

(b) The expected number of failures in the 8-year period.

(c) The regret table, and the optimum choice on the basis of least expected regret criterion.

(d) EVPI. **[Poona (M.B.A.) 96]**

[**Hint.** The cost functions can be written as

$$C = \begin{cases} 4000\ S \quad \text{when} \quad F < S \\ 4000\ S + 18{,}000\ (F - S), \quad \text{when} \quad F > S \end{cases}$$

Event (No. of failures)	*Prob.*	*Opportunity cost (in '000 Rs.) Act (No. of spares)*						*Regret values (in '000 Rs.) Act (No. of Spares)*					
		A_1 (0)	A_2 (1)	A_3 (2)	A_4 (3)	A_5 (4)	A_6 (5)	A_1 (0)	A_2 (1)	A_3 (2)	A_4 (3)	A_5 (4)	A_6 (5)
E_1 (0)	0·1	0	4	8	12	16	20	0	4	8	12	16	20
E_2 (1)	0·2	18	4	8	12	16	20	14	0	4	8	12	16
E_3 (2)	0·3	36	22	8	12	16	20	28	14	0	4	8	12
E_4 (3)	0·2	54	40	26	12	16	20	42	28	14	0	4	8
E_5 (4)	0·1	72	58	44	30	16	20	56	42	28	14	0	4
E_6 (5)	0·1	90	76	62	48	34	20	70	56	42	28	14	0
Column minima		0	4	8	12	16	20	Expected regret :					
Column maxima		90	76	62	48	34	20	32·2	20	11·4	8·2	8·6	10·8
Simple average cost		45	34	26	21	19	20						
Expected cost		41·4	29·2	20·6	17·4	17·8	20						

(a) No. of units on the basis of different principles :

(i) *Minimax.* The maximum values in each of the columns are indicated by the column maxima row. The minimum of these being 20, the decision on the basis of the maximum rule would be to buy 5 spare parts.

(ii) *Minimum.* From the minimum values contained in the row entitled column minima, it may be observed that the least value is equal to zero. Thus the optimum number of spare parts is nil.

(iii) *Laplace Principle.* According to this principle, the different events, E_1's are assumed to be equally probable. Thus, the decision is taken on the basis of the simple average cost values (determined without using the given probability values). Since the simple average cost is the minimum for A_5, it follows that the optimum number of spares, according to the Laplace rule, is 4.

(iv) *Expected cost Principle.* From the table above, we observe that the minimum value appears against the strategy A_4. Thus, according to the expectation principle, the optimal policy is to store 3 spare parts, the expected cost being Rs. 17·4 thousand.

(b) Expected number of failures in the 8 years period

$$E(F) = \sum_{i=1}^{6} P_i E_i = 0{\cdot}1 \times 0 + 0{\cdot}2 \times 1 + 0{\cdot}3 \times 2 + 0{\cdot}2 \times 3 + 0{\cdot}1 \times 4 + 0{\cdot}1 \times 5 = 2{\cdot}3$$

(c) The regret table can be derived from the table above, for it, the least cost value in each row would be subtracted from other values. The difference represent the regret value. Table below depicts in the 8-year period.

Since the expected regret for strategy A_4 is the least, the optimum policy according to this criterion, as in case of the expected cost principle, is to buy 3 spare parts.

(iv) The expected value of perfect information, EVPI, when the pay-off matrix indicates cost, is defined as follows :

$$\text{EVPI} = \begin{pmatrix}\text{Expected cost with}\\ \text{optimum policy}\end{pmatrix} - \begin{pmatrix}\text{Expected cost with}\\ \text{perfect information}\end{pmatrix} = 17{\cdot}4 - 9{\cdot}2 = 8{\cdot}2 \text{ ('000 Rs.)}$$

30. A cycle dealer finds that the cost of holding a cycle in stock for a week is Rs. 10. Customers who cannot obtain new cycles immeidately tend to go to other dealers, and he estimates that for every customer who cannot get immediate delivery, he loses an average of Rs. 80. For one particular model of cycle, the probabilities of a demand of 0, 1, 2, 3, 4 and 5 cycles in a week are 0·05, 0·10, 0·20, 0·30, 0·20 and 0·15 respectively. How many cycles should the dealer keep in stock per week ? (Assume that there is no time lag between ordering and delivery). **[Delhi (M.B.A.) March 99]**

31. Sameer lives off campus and has just missed the bus that could have taken him to campus for his 10 a.m. test. It is now 9·45 a.m. and Sameer has several options available to get him to campus : Waiting for the next bus, walking, riding his bike, or driving his car. The bus is scheduled to arrive in 10 minutes, and it will take Sameer exactly 20 minutes to get to his test from the time he gets on the bus. However, there is 0·2 chance that the bus will be 5 minutes early, a 0·3 chance that the bus will be 5 minutes late.

If Sameer walks, there is 0·8 chance he will get to test in 30 minutes, and 0·2 chance he will get there in 35 minutes.

If Sameer rides his bike, he will get to the test in 25 minutes with probability 0·5, 30 minutes with probability 0·4, and there is 0·1 chance of a flat tyre, causing him to take 45 minutes.

If Sameer drives his car to campus, he will take 15 minutes to get to campus, but the time needed to park his car and get to his test is given below :

Time to park and arrive (in min.)	10	15	20	25
Probability	0·30	0·45	0·15	0·10

Assuming that Sameer wants to minimize his expected late time in getting to his test, draw the decision tree and determine his best option. **[AIMA, P.G. Dip. in in Management) June 97]**

[Hint.

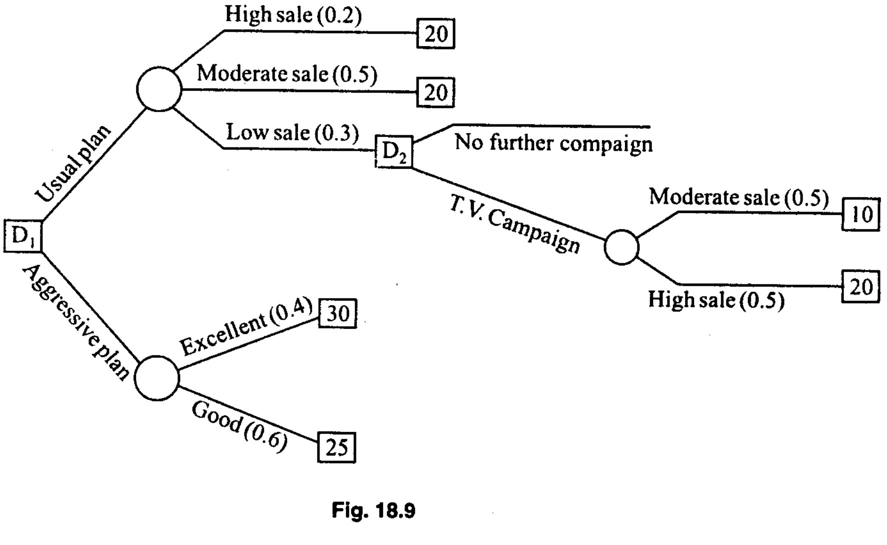


Fig. 18.9

32. What are the advantages of simulation ? **[IGNOU 99 (Dec.)]**

33. A decision problem has been expressed in the following pay-off table :

(a) What is the minimum pay-off action ?

(b) What is the minimum opportunity loss function ?

[Bhubneshwar (IT) 2004]

		Outcome		
		I	II	III
Action	A	10	20	26
	B	30	30	60
	C	40	30	20

OBJECTIVE QUESTIONS

1. A type of decision-making environment is

(a) certainty. (b) uncertainty. (c) risk. (d) all of the above.

2. Decision theory is concerned with
   (a) methods of arriving at.
   (b) selecting optimal decision in sequential manner optimal decision.
   (c) analysis of information that is available.
   (d) all of the above.
3. Which of the following criteria is not used for decision-making under uncertainty?
   (a) Maximin. (b) Maximax.
   (c) Minimax. (d) Minimize the expected loss.
4. Which of the following criteria is not applicable to decision-making under risk?
   (a) Maximize expected return.
   (b) Maximize return.
   (c) Minimize expect regret.
   (d) Knowledge of likelihood occurrence of each state of nature.
5. The minimum expected opprtunity loss (EOL) is
   (a) equal to EVPI. (b) minimum regret. (c) equal to EMV. (d) both (a) and (b).
6. The expected value of perfect information (EVPI) is
   (a) equal to expected regret of the optimal decision under risk.
   (b) the utility of additional information.
   (c) maximum expected opportunity loss.
   (d) none of the above.
7. The value of the coefficient of optimism (α) is needed while using the criterion of
   (a) equally likely. (b) maximin. (c) realism. (d) minimax.
8. The decision-maker's knowledge and experience may influence the decision-making process when using the criterion of
   (a) maximax. (b) minimax regret. (c) realism. (d) maximin.
9. The difference between the expected profit under conditions of risk and the expected profit with perfect information is called
   (a) expected value of perfect information. (b)expected marginal loss.
   (c) expected opportunity loss. (d) none of the above.
10. The concept of utility is used to
    (a) measure the utility of money. (b) take into account aversion of risk.
    (c) both (a) and (b). (d) none of the avove.

Answers

1. (d) **2.** (d) **3.** (d) **4.** (d) **5.** (d) **6.** (a) **7.** (c) **8.** (c) **9.** (a) **10.** (c).

* * *

UNIT 4

QUANTITATIVE TECHNIQUES IN OPERATIONS RESEARCH

CONTAINING :

19

THEORY OF GAMES
(Competition Strategies)

19.1. INTRODUCTION

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an *election* have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing compaigns by competing business firms, etc.

What should be the bid to win a big Government contract in the pace of competition from several contractors ? Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must *win* and somebody must *lose.*

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the ***'minimax (maximin)*** criterion' of **J. Von Neumann** (called the father of game theory). This criterion implies the assumption of rationality from which it is argued that each player will act so as **to *maximize his minimum gain*** or ***minimize his** maximum loss.* The difficulty lies in the deduction from the assumption of 'rationality' that the other player will maximize his minimum gain. There is no agreement even among game theorists that rational players should so act. In fact, rational players do not act apparently in this way, or in any consistent way. Therefore, game theory is generally interpreted as an "as if" theory, that is, as if rational decision maker (player) behaved in some well defined (but arbitrarily selected) way, such as *maximizing the minimum gain.*

The game theory has only been capable of analysing very simple competitive situations. Thus, there has been a great gap between what the theory can handle and most actual competitive situations in industry and elsewhere. So the primary contribution of game theory has been its concepts rather than its formal application to solving real problems.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).

*The set of rules defines the **game**. Going through the set of rules once by the participants defines a **play**.*

19.2. CHARACTERISTICS OF GAME THEORY

There can be various types of games. They can be classified on the basis of the following characteristics.

(i) ***Chance of strategy*** : If in a game, activities are determined by skill, it is said to be a ***game of strategy;*** if they are determined by chance, it is a *game of chance.* In general, a game may involve game of strategy as well as a game of chance. In this chapter, simplest models of games of strategy will be considered.

(ii) ***Number of persons*** : A game is called an n-person game if the number of persons playing is n. The person means an individual or a group aiming at a particular objective.

(*iii*) ***Number of activities*** : These may be *finite* or *infinite*.

(*iv*) ***Number of alternatives (choices) available to each person*** in a particular activity may also be finite or infinite. A *finite* game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be *infinite*.

(*v*) ***Information to the players about the past activities of other players*** is completely available, partly available, or not available at all.

(*vi*) ***Payoff*** : A quantitative measure of satisfaction a person gets at the end of each play is called a *payoff*. It is a real-valued function of variables in the game. Let v_i be the payoff to the player P_i, $1 \leq i \leq n$, in an n-person game. If $\Sigma_{i=1}^{n} v_i = 0$, then the game is said to be a *zero-sum game*.

In this chapter, we shall discuss *rectangular games (also called two-person zero-sum)* only.

Q. 1. Write a short note on characteristics of game theory. [Rewa M.Sc. (Math) 93]

2. What is game theory ? List out the assumptions made in the theory of games. [JNTU 2003, 02]

19.3. BASIC DEFINITIONS

1. Competitive Game. A competitive situation is called a ***competitive game*** if it has the following four properties : **[JNTU (B. Tech.) 2004, 03; Meerut 2002]**

(*i*) There are finite number (n) of competitors (called players) such that $n \geq 2$. In case $n = 2$, it is called a **two-person game** and in case $n > 2$, it is referred to as an ***n*-person game.**

(*ii*) Each player has a list of finite number of possible activities (the list may not be same for each player).

(*iii*) A play is said to *occur* when each player chooses one of his activities. The choices are assumed to be made simultaneously, *i.e.* no player knows the choice of the other until he has decided on his own.

(*iv*) Every combination of activities determines an outcome (which may be points, money or any thing else whatsoever) which results in a gain of payments (+ ve, – ve or zero) to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

2. Zero-sum and Non-zero-sum Games. Competitive games are classified according to the number of players involved, *i.e.* as a *two person game, three person game,* etc. Another important distinction is between *zero-sum games* and *nonzero-sum* games. If the players make payments only to each other, *i.e.* the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be *zero-sum*.

Mathematically, suppose an n-person game is played by n players $P_1, P_2, \dots, P_n$ whose respective pay-offs at the end of a play of the game are $v_1, v_2, \dots v_n$ then, the game will be called zero-sum if $\sum_{i=1}^{n} v_i = 0$ at each play of the game. **[JNTU (Mech. & Prod.) 2004]**

A game which is not zero-sum is called a *nonzero-sum game*. Most of the competitive games are zero-sum games. An example of a nonzero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

3. Strategy. A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rules (programmes) that specifies which of the available course of action he should make at each play. This strategy may be of two kinds : **[JNTU (B. Tech.) 2004, 03; Meerut 2002; IGNOU 2001, 2000, 98, 97]**

(*i*) ***Pure Strategy.*** : If a player knows exactly what the other player is going to do, a *deterministic* situation is obtained and objective function is to maximize the gain. *Therefore, the pure strategy is a decision rule always to select a particular course of action.* **[Meerut 2002]**

A pure strategy is usually represented by a number with which the course of action is associated.

(*ii*) ***Mixed Strategy.*** **[Agra 92; Kerala (Stat.) 83]** : If a player is guessing as to which activity is to be selected by the other on any particular occasion, a *probabilistic* situation is obtained and objective function is to maximize the *expected gain*. **[Meerut 2003, 02]**

Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Mathematically, a mixed strategy for a player with m (≥ 2) possible courses of action, is denoted by the set S of m non-negative real numbers whose sum is unity, representing probabilities with which each course of action is chosen. If x_i ($i = 1, 2, 3, ..., m$) is the probability of choosing the course i, *then*

$$S = (x_1 , x_2 , x_3 , \dots , x_m) \qquad ...(19\cdot1)$$

subject to the conditions

$$x_1 + x_2 + x_3 + \dots + x_m = 1 \qquad ...(19\cdot2)$$

and

$$x_1 \geq 0 , x_2 \geq 0 , x_3 \geq 0 , \dots , x_m \geq 0 . \qquad ...(19\cdot3)$$

Note. It should be noted that if some $x_i = 1$, ($i = 1, 2, ..., m$) and all others are zero, the player is said to use a pure strategy. Thus, the pure strategy is a particular case of mixed strategy.

4. Two-Person, Zero-Sum (*Rectangular*) Games. A game with only two players (say, *Player A* and *Player B*) is called a *'two-person, zero-sum game'* if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero. **[JNTU (B. Tech.) 2003]**

Two-person, zero-sum games, are also called *rectangular games* as these are usually represented by a payoff matrix in rectangular form. **[Meerut (OR) 2003]**

5. Payoff Matrix. Suppose the player A has m activities and the player B has n activities. Then a payoff matrix can be formed by adopting the following rules : **[Meerut (OR) 2003; JNTU (B. Tech) 97]**

(i) Row designations for each matrix are activities available to player A.

(ii) Column designations for each matrix are activities available to player B.

(iii) Cell entry 'v_{ij} ' is the payment to player A in A's payoff matrix when A chooses the activity i and B chooses the activity j.

(iv) With a 'zero-sum, two person game', the cell entry in the player B's payoff matrix will be negative of the corresponding cell entry 'v_{ij} ' in the player A's payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.

Table 19·1. The player *A*'s payoff matrix

Player *B*

		1	2	...	j	...	n
	1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
	2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
Player *A*	:	:	:		:		:
	i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
	:	:	:		:		:
	m	v_{m1}	v_{m2}	...	v_{mj}	...	v_{mn}

Table 19·2. The player *B*'s payoff matrix

Player *B*

		1	2	...	j	...	n
	1	$-v_{11}$	$-v_{12}$	...	$-v_{1j}$	...	$-v_{1n}$
	2	$-v_{21}$	$-v_{22}$	...	$-v_{2j}$	...	$-v_{2n}$
Player *A*	:	:	:		:		:
	i	$-v_{i1}$	$-v_{i2}$	...	$-v_{ij}$	...	$-v_{in}$
	:	:	:		:		:
	m	$-v_{m1}$	$-v_{m2}$	...	$-v_{mj}$	...	$-v_{mn}$

Note. Further, there is no need to write the B's payoff matrix as it is just the –ve of A's payoff matrix in a *zero-sum two-person game*. Thus, if 'v_{ij}' is the gain to A, then '$-v_{ij}$' will be the gain to B.

In order to make the above concepts a clear, consider the coin matching game involving two players only. Each player selects either a head H or a tail T. If the outcomes match (H, H or T, T), A wins Re 1 from B , otherwise B wins Re 1 from A . This game is a two-person zero-sum game, since the winning of one player is taken as losses for the other. Each has his choices between two *pure* strategies (H or T). This yields the following (2 × 2) payoff matrix to player A.

Table 19.3.

B

		H	T
A	H	+1	–1
	T	–1	+1

It will be shown later that the optimal solution to such games requires each player to play one pure strategy or a mixture of pure strategies.

Q. 1. State the four properties which a competitive situation should have, if it is to be called a competitive game.

2. What is the problem studied in game theory ? **[IGNOU (B.Com) 91]**

3. Define :
   (i) Competitive game
   (ii) Pure strategies
   (iii) Mixed strategies

(iv) Two-person, Zero-sum (or Rectangular games) **[Agra 94]**
(v) Payoff matrix **[IGNOU 2001, 2000, 98, 97; JNTU (B. Tech) 97]**

4. (i) Explain zero-sum two-person game giving suitable example.
(ii) What is a zero-sum two-person game ? **[IGNOU 2001, 2000, 98, 97; Agra 92]**
(iii) Explain the difference between pure strategy and mixed strategy.

19.4. MINIMAX (MAXIMIN) CRITERION AND OPTIMAL STRATEGY

The 'minimax criterion of optimality' *states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an* ***optimal strategy.***

Example 1. *Consider (two-person, zero-sum) game matrix which represents payoff to the player A. Find the optimal strategy, if any.* [***See Table 19·4***]

Table 19·4

Saddle Point		B				
		I	II	III	Row minimum	
	I	–3	–2	$\boxed{-3}$	–3	
A	II	2	$\boxed{0}$	2	$\boxed{0}$	Maximin Value ($\underline{v}$)
	III	$\boxed{5}$	–2	$\boxed{-4}$	–4	
Column maximum		$\boxed{5}$	$\boxed{0}$	$\boxed{6}$		

Minimax Value ($\bar{v}$)

Solution. The player A wishes to obtain the largest possible 'v_{ij}' by choosing one of his activities (I, II, III), while the player B is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III). The player A is called the ***maximizing player*** and B the ***minimizing player.***

If the player A chooses the Ist activity, then it could happen that the player B also chooses his Ist activity. In this case the player B can guarantee a gain of at least – 3 to player A , *i.e.*

$$\min\{-3, -2, 6\} = -3$$

Similarly, for other choices of the player A , *i.e.* II and III activities, B can force the player A to get only 0 and – 4, respectively, by his proper choices from (I, II, III), *i.e.*

$$\min\{2, 0, 2\} = 0 \quad \text{and} \quad \min\{5, -2, -4\} = -4$$

The minimum value in each row guaranteed by the player A is indicated by 'row minimum' in (*Table 19·4*). The best choice for the player A is to maximize his least gains –3, 0, – 4 and opt II strategy which assures at most the gain 0, *i.e.*

$$\max\{-3, 0, -4\} = \boxed{0}$$

In general, the player A should try to maximize his least gains or to find out "$\max_i \min_j v_{ij}$"

Player B, on the other hand, can argue similarly to keep A's gain the minimum. He realizes that if he plays his Ist pure strategy, he can loose no more than 5 = max {– 3, 2, 5} regardless of A 's selections. Similar arguments can be applied for remaining strategies II and III. Corresponding results are indicated in *Table 19·4* by *'column maximum'*. The player B will then select the strategy that minimizes his maximum losses. This is given by the strategy II and his corresponding loss is given by

$$\min\left\{\boxed{5}, \boxed{0}, \boxed{6}\right\} = \boxed{0}$$

The player A's selection is called the ***maximin strategy*** and his corresponding gain is called the ***maximin value*** or ***lower value*** ($\underline{v}$) of the game. The player B's selection is called the ***minimax value*** or ***upper value*** ($\bar{v}$) of the game. The selections made by player A and B are based on the so called ***minimax*** (or ***maximin***) ***criterion***. It is seen from the governing conditions that the minimax (upper) value $\bar{v}$ is greater than or equal to the maximin (lower) value $\underline{v}$ (see *Theorem 19·1*). In the case where equality holds *i.e.*,

$$\max_i \min_j v_{ij} = \min_j \max_i v_{ij} \quad \text{or} \quad \underline{v} = \bar{v}, \qquad ...(19·4)$$

the corresponding pure strategies are called the *'optimal'* strategies and the game is said to have a ***saddle point.*** It may not always happen as shown in the following example.

Note. For convenience, the minimum values are shown by 'O' and maximum values by '☐' in Table 19·4.

Q. 1. What is a competitive situation called a game ? What is the maximin criterion of optimality ?

2. What is a game in the game theory ? What are the properties of a game ? Explain the best (optimal) strategy on the basis of minimax criterion of optimality.

3. What are the assumptions made in the theory of games?

4. Explain Maxi-Min and Mini-Max principle used in Game Theory. **[JNTU (B. Tech) 2000; Agra 94]**

Example 2. *Consider the following game :*

			B	
	i \ *j*	1	2	3
	1	3	−4	8
A	2	−8	5	−6
	3	6	−7	6

As discussed in ***Example 1***, $\max_i \min_j v_{ij} = 4, \quad \min_j \max_i v_{ij} = 5.$

Also, $\max_i \min_j v_{ij} < \min_j \max_i v_{ij}$

Such games are said to be the games ***without saddle point.***

Example 3. *Find the range of values of p and q which will render the entry (2, 2) a saddle point for the game :* **[JNTU (B. Tech.) 2003]**

		Player *B*	
	2	4	5
Player *A*	10	7	*q*
	4	*p*	6

Solution. First ignoring the values of p and q determine the maximin and minimax values of the payoff matrix as below :

Since the entry (2, 2) is a saddle point, maximin value $\underline{v} = 7$, minimax value $\overline{v} = 7$.

This imposes the condition on p as $p \le 7$ and on q as $q \ge 7$. Hence the range of p and q will be $p \le 7$, $q \ge 7$.

Theorem 19·1. *Let $\{v_{ij}\}$ be the payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then $\overline{v} \ge \underline{v}$. That is,* $\min_j [\max_i \{v_{ij}\}] \ge \max_i [\min_j \{v_{ij}\}]$ **[Meerut (Stat.) 90]**

Proof. We have, $\max_i \{v_{ij}\} \ge v_{ij}$ for any j, and $\min_i \{v_{ij}\} \le v_{ij}$ for any i.

		Player B			
		B_1	B_2	B_3	Row Min..
	A_1	2	4	5	2
Player A	A_2	$\boxed{⑩}$	$\boxed{7}$	q	7
	A_3	4	ⓟ	$\boxed{6}$	4
Column Max.		10	7	6	

Let the above maximum be attained at $i = i^*$ and the minimum be attained at $j = j^*$. So

$$v_{i^*j} \ge v_{ij} \ge v_{ij^*} \qquad \text{for any } i \text{ and } j.$$

This implies that

$$\min_j \{v_{i^*j}\} \ge v_{ij} \ge \max_i \{v_{ij^*}\} \qquad \text{for any } i \text{ and } j \,.$$

Hence $$\min_j [\max_i \{v_{ij}\}] \ge \max_i [\min_j \{v_{ij}\}] \quad \text{or} \quad \overline{v} \ge \underline{v}\,.$$

19·5. SADDLE POINT, OPTIMAL STRATEGIES AND VALUE OF THE GAME

Definitions :

Saddle Point. *A **saddle point** of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column.* **[JNTU (MCA III) 2004, 97; Meerut 2003, 02; IGNOU 99, 98]**

Mathematically, *if a payoff matrix* $\{v_{ij}\}$ *is such that* $\max\limits_{i}\,[\min\limits_{j}\,\{v_{ij}\}] = \min\limits_{j}\,[\max\limits_{i}\,\{v_{ij}\}] = v_{rs}$ (say), *then the matrix is said to have a saddle point* (r, s).

Optimal Strategies. *If the payoff matrix* $\{v_{ij}\}$ *has the saddle point* (r, s), *then the players (A and B) are said to have* ***rth and sth optimal strategies****, respectively.* **[Meerut (OR) 2003; JNTU 97]**

3. Value of Game. *The payoff* (v_{rs}) *at the saddle point* (r, s) *is called the* ***value of game*** *and it is obviously equal to the maximin* ($\underline{v}$) *and minimax value* ($\overline{v}$) *of the game.* **[IGNOU 99, 98]**

A game is said to be a ***fair game*** if $\overline{v} = \underline{v} = 0$. A game is said to be ***strictly determinable*** if $\overline{v} = v = \underline{v}$.

Note. A saddle point of a payoff matrix is, sometimes, called the equilibrium point of the payoff matrix.

In *Example 1*, $\underline{v} = \overline{v} = 0$. This implies that the game has a saddle point given by the entry (2, 2) of payoff matrix. The value of the game is thus equal to zero and both players select their strategy as the optimal strategy. In this example, it is also seen that no player can improve his position by other strategy.

In general, a matrix need not have a saddle point as defined above. Thus, these definitions of optimal strategy and value of the game are not adequate to cover all cases so need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such generalization. These theorems are presented in ***Sec. 19·24.***

Rules for Determining a Saddle Point :

1. ***Select the minimum element of each row of the payoff matrix and mark them by 'O'.***
2. ***Select the greatest element of each column of the payoff matrix and mark them by '□ '.***
3. ***If there appears an element in the payoff matrix marked by 'O' and '□ ' both, the position of that element is a saddle point of the payoff matrix.***

Q. 1. Define : (i) Competitive Game, (ii) Payoff matrix, (iii) Pure and mixed strategies, (iv) Saddle point, (v) Optimal strategies, and (vi) Rectangular game **[JNTU 2000; Kanpur M.Sc. (Maths.) 93]**

2. Explain "best strategy" on the basis of minimax criterion of optimalities.
3. Describe the maximin principle of game theory. What do you understand by pure strategies and saddle point. **[SJMIT (BE Mech.) 2002; Punjabi (M.B.A.) 90]**
4. Define saddle point and the value of game with examples. **[Meerut 2002; GNDU (B. Com.) 91]**
5. Define saddle point. Is it necessary that a said game should always possess a saddle point ?
6. State the rules for detecting a saddle point.
7. What is 'strictly determined game' ? When a game is said to be determinable ?
8. Write short notes on the following : (i) Pure Strategy, (ii) Mixed Strategy, (iii) Max-Min Criterion.
9. Let $A = \{a_{ij}\}$ be an $m \times n$ payoff matrix for a zero-sum two-person game. Define a Saddle point for matrix A and show that the value of the game is equal to the saddle value.
10. Differentiate between strictly determinable game and non-determinable games. **[JNTU (Mech. & Prod.) 2004]**

EXAMINATION PROBLEMS

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games : **[JNTU (B. Tech.) 2003]**

(i)
$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix} \end{array}$$
[Ans. Not fair, $v = 1$]

(ii)
$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$
[Ans. Not fair, (I, II), $v = -5$]

2. Consider the game G with the following payoff matrix :

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix} \end{array}$$

(i) Show that G is strictly determinable whatever μ may be. (ii) Determine the value of G.
[Ans. (I, I), $v = 2$] **[Jodhpur M.Sc. (Math.) 92]**

3. Find out whether there is any saddle point in the following problem :

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \end{array}$$

[Ans. Saddle point does not exist.]

4. For the game with payoff matrix :

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

determine the best strategies for players A and B and also the values of the game for them. Is this game (*i*) fair (*ii*) strictly determinable ?
[Ans. (I, II), $v = -2$.
Game is strictly determinable and not fair.]

19.6. SOLUTION OF GAMES WITH SADDLE POINTS

To obtain a solution of a rectangular game, it is feasible to find out :

(i) the best strategy for player A *(ii)* the best strategy for player B, and *(iii)* the value of the game (v_{rs}).

It is already seen that the best strategies for players A and B will be those which correspond to the row and column, respectively, through the saddle point. The value of the game to the player A is the element at the saddle point, and the value to the player B will be its negative.

19.7. ILLUSTRATIVE EXAMPLES

Example 4. *Player A can choose his strategies from* $\{A_1, A_2, A_3\}$ *only. while B can choose from the set* (B_1, B_2) *only. The rules of the game state that the payments should be made in accordance with the selection of strategies :*

Strategy Pair Selected	*Payments to be Made*	*Strategy Pair Selected*	**Payments to be Made**
(A_1, B_1)	*Payer A Pays Re. 1 to player B*	(A_2, B_2)	*Player B pays Rs 4 to player A*
(A_1, B_2)	*Player B pays Rs. 6 to player A*	(A_3, B_1)	*Player A pays Rs 2 to player B*
(A_2, B_1)	*Player B pays Rs 2 to player A*	(A_3, B_2)	*Player A pays Rs. 6 to player B*

What strategies should A and B play in order to get the optimum benefit of the play ?

Solution. With the help of above rules the following payoff matrix is constructed :

The payoffs marked 'O' represent the minimum payoff in each row and those marked '☐' represent the maximum payoff in each column of the payoff matrix.

Obviously, the matrix has a saddle point at position (2, 1) and the value of the game is 2.

Thus, the optimum solution to the game is given by :

(i) the optimum strategy for player A is A_2 ; *(ii)* the optimum strategy for player B is B_1 ; and

(iii) the value of the game is Rs. 2 for player A and Rs. (– 2) for player B.

Also, since $v \neq 0$, the game is not fair, although it is strictly determinable.

		Player B	
		B_1	B_2
	A_1	$\textcircled{-1}$	$\boxed{6}$
Player A	A_2	$\boxed{\textcircled{2}}$	4
	A_3	-2	$\textcircled{-6}$

Example 5. *The payoff matrix of a game is given. Find the solution of the game to the player A and B.*

$$\begin{array}{cc} & B \\ & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ A\ \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix} \end{array}$$

[JNTU (MCA III) 2004, (B. Tech.) 2000, 99]

Solution. First find out the saddle point by encircling each row minima and putting squares around each column maxima.

The saddle point thus obtained is shown by having a circle and square both (*Table 19·5*)

Table 19·5

OPTIMUM STRATEGY FOR B

		I	II	III	IV	V	ROW MINIMUM	
	I	$\textcircled{-2}$	0	0	$\boxed{5}$	3	$\textcircled{-2}$	
OPTIMUM STRATEGY FOR A	II	3	2	$\boxed{\textcircled{1}}$	2	2	$\boxed{\textcircled{1}}$	MAXIMIN VALUE (v)
	III	$\textcircled{-4}$	-3	0	-2	$\boxed{6}$	-4	
	IV	$\boxed{5}$	$\boxed{3}$	-4	2	$\textcircled{-6}$	$\textcircled{-6}$	
COLUMN MAXIMUM		$\boxed{5}$	$\boxed{3}$	$\textcircled{\boxed{1}}$	$\boxed{5}$	$\boxed{6}$		

Minimax Value ($\bar{v}$)

Hence, the solution to this game is given by, *(i)* the best strategy for player *A* is 2nd; *(ii)* the best strategy for player *B* is 3rd; and *(iii)* the value of the game is 1 to player *A* and – 1 to player *B*.

Example 6. *Solve the game whose payoff matrix is given by*

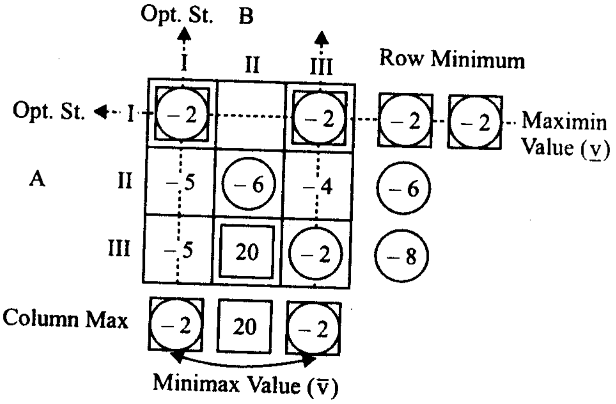
$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \left[\begin{array}{ccc} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{array}\right] \end{array}$$

[Kanpur M.Sc. (Math.) 96; Rewa (M.P.) 93]

Solution. *Table* 19·6 may be formed as explained earlier.

This game has two saddle points in positions (1, 1) and (1, 3). Thus, the solution to this game is given by,

(i) the best strategy for the player *A* is I, *(ii)* the best strategy for the player *B* is either I or III, *i.e.* the player *B* can use either of the two strategies (I, III), and *(iii)* the value of the game is – 2 for player *A* and + 2 for player *B*.



EXAMINATION PROBLEMS

Find the saddle point (or points) and hence solve the following games :

1. Player *B*

$$\text{Player A}\;\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \left[\begin{array}{ccc} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{array}\right] \end{array}$$

[Ans. (A_2, B_2), $v = 5$]

2. *B*

$$A\;\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{array}\right] \end{array}$$

[Ans. (A_2, B_2), $v = 4$]

3. *B*

$$A\;\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \left[\begin{array}{cccc} -5 & 2 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ 4 & -2 & 0 & -5 \end{array}\right] \end{array}$$

[Ans. (II, III), $v = 4$]

4. *B*

$$A\;\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}\begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \left[\begin{array}{ccccc} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 4 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{array}\right] \end{array}$$

[Ans. (II, III), $v = 4$]

5.

$$\begin{array}{c} \\ \text{I} \\ \text{II} \end{array}\begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \left[\begin{array}{ccc} 6 & 8 & 6 \\ 4 & 12 & 2 \end{array}\right] \end{array}$$

[Ans. (I, I), (I, III), $v = 6$]

6.

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \left[\begin{array}{ccc} 3 & 0 & -3 \\ 2 & 3 & 1 \\ -4 & 2 & -1 \end{array}\right] \end{array}$$

[Ans. (R_2, C_3), $v = 1$]

7. Solve the game whose payoff matrix is given by

Player *B*

$$\text{Player } A\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 1 & 3 & -2 \end{array}\right]$$

[Agra 98]

[Ans. (I, I) or (I, III), $v = 1$ for *A*, $v = -1$ for *B*]

8. For the following payoff matrix for firm *A*, determine the optimal strategies for both the firms and the value of the game (using maximin-minimax principle) :

Firm *B*

$$\text{Firm } A\left[\begin{array}{ccccc} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{array}\right]$$

[Ans. (III, III), $v = 6$ for *A*]

9. Solve the games whose payoff matrices are given below :

(a) Player *B*

$$\text{Player A}\;\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \left[\begin{array}{ccc} -3 & -1 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{array}\right] \end{array}$$

(b) Player *B*

$$\text{Payer } A\left[\begin{array}{ccc} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{array}\right]$$

(c)

$$\left[\begin{array}{cccc} -5 & 5 & 0 & 7 \\ 2 & 6 & 1 & 8 \\ -4 & 0 & 1 & -3 \end{array}\right]$$

[Ans. (a) (A_2, B_2), $v = 0$; (b) (A_2, B_2), $v = 5$; (c) (A_2, A_3), $v = 1$]

[Kanpur 2000]